Department of Mathematics
University of Utah
Real and Complex Analysis Preliminary Examination

Instructions: Do seven problems and list on the front of your blue book the seven problems to be graded. Do at least three problems from each part. Two correct solutions from each section will represent a passing exam.

Part A:

Let λ denote Lebesgue measure on $\mathbb{R}$. Let $\mathcal{F}$ denote the Fourier transform.

Problem 1. If $f \in L^1(\lambda)$ then $\lim_{c\to 1} \int |f(cx) - f(x)| d\lambda = 0$.

Problem 2. If $f : \mathbb{R} \to \mathbb{R}$ is Lipshitz and A is λ-measurable then $f(A)$ is measurable. Recall that a function is Lipschitz if there exists C so that $d(fx, fy) \le Cd(x, y)$ for all $x, y \in \mathbb{R}$.

Problem 3. Show that $f \in L^1$ then $\lim_{x\to\infty} \mathcal{F}(f)(x) = 0$.

Problem 4. Show that $\{\bar{v} \in \ell^p : |v_i| \le 2^{-i}\}$ is compact for all $p \ge 1$.

Problem 5. Prove that if μ is a non-atomic Borel probability measure on $\mathbb{R}$ then $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = \mu((-\infty, t))$ is continuous.

Part B:

Problem 1. Evaluate the integral by the method of residue:

$$\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx,$$

for $a \in \mathbb{R}$.

Problem 2. Let $f(z)$ be analytic in a deleted neighborhood of $z = a$ and $\lim_{z \to a}(z - a)f(z) = 0$. Does it mean that $f(z)$ has removable singularity at a? If so, prove it. If not, give a counterexample.

Problem 3. If $f(z)$ is analytic in $|z| \leq 1$ and satisfies $|f| = 1$ on $|z| = 1$, show that $f(z)$ is rational.

Problem 4. Prove that in any region Ω the family of analytic functions whose values lie on an open half plane is normal.

Problem 5. Let $\wp(z)$ be the Weierstrass $\wp$ function. Show that any even elliptic function $f(z)$ with periods ω_1, ω_2 can be expressed in the form

$$(\text{Constant}) \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}$$

for some Constant, $a_k, b_k \in \mathbb{C}$, provided that 0 is neither a zero or a pole. What is the corresponding form if $f(z)$ either vanishes or has poles at $z = 0$?