DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN ANALYSIS
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Instructions: Do seven problems with at least three (3) problems from section A and three (3) problems from section B. You need at least two problems completely correct from each section to pass. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points.

1. Let $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = e^{-x}$. Explain why f is Lebesgue integrable and compute its integral.

2. Let V be a normed space and U a closed subspace. For every $x \in V$, let
$$||x + U|| = \inf\{||x + y|| \mid y \in U\}.$$
   (a) Prove that $||x + U||$ is a norm on V/U.
   (b) If V is complete, so is V/U.

3. Let $T : V \to U$ be a bounded map between two normed spaces. Let $T^* : U^* \to V^*$ be defined by $T^*(f) = f \circ T$ for all $f \in U^*$ (the adjoint map). Prove that $||T^*|| = ||T||$.

4. Let V be a Banach space, and $T : V \to V$ a bounded linear map. Assume that for every $v \in V$ there exists a non-negative integer n such that $T^n(v) = 0$. Prove that there exists an integer n such that $T^n(v) = 0$ for all $v \in V$.

5. Let $X = [0, 1]$ with the usual measure. Let $p \geq 1$. For $n = 1, 2, \ldots$, let $f_n = n^{1/p} \cdot \chi_{[0,1/n]}$. Prove that the sequence f_n converges weakly to 0 in $L^p([0, 1)]$ if and only if $p > 1$. (Use that $L^p([0, 1)]^* \cong L^q([0, 1)]$.)

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

Notation: Let $\mathbb{H}$ be the upper half plane and $\mathbb{D}$ be the (open) unit disk.

6. Let $a_j \in \mathbb{C}$ satisfy $|a_j| \leq 1$ for all $j \geq 2$. Show that $f : B(0, \frac{1}{100}) \to \mathbb{C}$ by $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ is injective.

7. Show that 0 is in the range of $e^z - z$.

8. Let $f : \mathbb{H} \to \mathbb{D}$ be a bijective holomorphic map. Are we guaranteed that $-f(\bar{z}) = -\overline{f(z)}$?

9. Use the methods of complex analysis to compute $\int_{-\infty}^{\infty} \frac{2}{(x^2+4)(x^2+1)} dx$.

10. If f, g are entire functions and $|f(z)| \leq |g(z)|$ for all z then there exists $c \in \mathbb{C}$ so that $g(z) = cf(z)$.