DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN ANALYSIS
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Instructions: Do seven problems with at least three (3) problems from section A and three (3) problems from section B. You need at least two problems completely correct from each section to pass. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points.

1. Let μ be the outer measure on $\mathbb{R}$ defined by

$$\mu(A) = \inf\{\sum_{i=1}^{\infty}(b_i - a_i) \mid A \subseteq \cup_{i=1}^{\infty}(a_i, b_i],\, a_i \leq b_i\}$$

for every $A \subset \mathbb{R}$. Prove that $\mu((a, b]) = b - a$. You may use that the length of a segment covered by finitely many segments is less than the sum of lengths of segments in the covering.

2. Let (X, μ) be a (not necessarily finite) measure space and $f \geq 0$ be a μ integrable function. Prove that, for every $\epsilon > 0$, there exists a set $E \subset X$ of finite measure such that

$$\int_E f d\mu > \int_X f d\mu - \epsilon.$$

3. Let $T : U \to V$ be a linear map between two Banach spaces such that, for any linear functional f on V, the composite $f \circ T$ is a continuous functional on U. Prove that T is continuous. Hint: use closed graph theorem.

4. Let $H = L^2(\mathbb{R}/\mathbb{Z})$, with the usual Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ of total measure 1. It is well know that this space has a Hilbert basis $e_n = e^{2\pi i n x}$, $n \in \mathbb{Z}$. Let $f \in H$ be given by $f(x) = x$ on $[0, 1)$. Compute $||f||_2^2$ directly and then using Parseval's equality.

5. Let H be a Hilbert space and $e_1, e_2, \ldots$ and an orthonormal basis. Prove that 0 is a weak limit of the sequence e_n. Prove that the unit ball B, $||x|| \leq 1$, is closed in the weak topology. Prove that the unit sphere S, $||x|| = 1$, is dense in B in the weak topology.

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

Notation: Let $\mathbb{D}$ be the (open) unit disk.

6. Show that all the singularities of $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \frac{1}{e^z+3z}$ have order 1. You do not need to find the singularities.

7. Either prove that there is no entire function f so that $f(n) = 0$ for all $n \in \mathbb{N}$ and $f(z) \neq 0$ for all $z \in \mathbb{C} \setminus \mathbb{N}$ or give an example of such a function with justification.

8. Prove or disprove: Let $\Omega = \{z : |Re(z)| < 1 \text{ and } |Im(z)| < 1\}$ if $f : \mathbb{D} \to \Omega$ is a biholomorphism so that $f(z)$ is real for all real $z \in \mathbb{D}$ and $f(0) = 0$ then $f(iz) = if(z)$.

9. How many zeros (with multiplicity) does $z^6 + 5z^4 + 1$ have in $B(0, 2)$?

10. State and prove Morera's Theorem.