Statistics Prelim Exam

January 2016

You need to collect 60 points to pass the exam. You will get 10 points for the solution of each correct question.

1. Let $X_1, \ldots, X_n$ be a random sample with density function $f(x) = 2e^{-2(x-\eta)}$ for $x > \eta$, and zero otherwise. We wish to test $H_0 : \eta = 4$ versus the alternative $H_A : \eta > 4$. We reject H_0 if $X_{(1)} = \min_{1 \le i \le n} X_i \ge 4 + 2/n$.

   (a) Compute the significance level of the test.

   (b) Compute the power function.

2. Let $X_1, \ldots, X_n$ be iid exponential(1) random variables. Find the joint pdf of the pair $(X_{(1)}, X_{(3)})$, where $X_{(i)}$ denotes the ith *order statistic*, i.e.

$$X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}.$$

3. Let X_1 and X_2 be independent with joint pdf $f(x) = x^{-2}$ for $1 \le x < \infty$ and 0 otherwise.

   (a) Find the joint pdf of $U = X_1 X_2$ and $V = X_2$.

   (b) Find the marginal pdf of U.

4. Assume that $X_1, \ldots, X_n$ are a random sample with common pdf $f(x; \theta) = (\theta + 1)x^{-\theta-2}$ for $x > 1$, and zero otherwise. Is the method of moments estimator for θ the same as the maximum likelihood estimator for θ?

5. Let $X_1, \ldots, X_n$ be iid with the Gamma(θ, κ) distribution, meaning that their common pdf is

$$f(x; \theta, \kappa) = \frac{1}{\theta^\kappa \Gamma(\kappa)} x^{\kappa-1} e^{-x/\theta}, \quad x > 0,$$

   with κ known but θ unknown. Derive a $100(1-\alpha)\%$ equal-tailed confidence interval for θ based on the sufficient statistic.

6. Let $X_1, X_2, \ldots, X_n$ be iid exponential(1) random variables and

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

be the sample mean. Find two sequences of numbers a_n and b_n such that

$$\frac{\log \bar{X}_n - a_n}{b_n}$$

converges in distribution to a standard normal random variable.

7. Suppose that $X_1, \ldots, X_n$ are iid from a BETA(θ_1, θ_2) distribution, meaning that the pdf is

$$f(x; \theta_1, \theta_2) = \frac{\Gamma(\theta_1 + \theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)} x^{\theta_1 - 1}(1-x)^{\theta_2 - 1}, \quad 0 < x < 1.$$

   Find jointly sufficient statistics for θ_1 and θ_2, and use them to find a UMVUEs for θ_1 and θ_2.

8. Let $Y_1 < Y_2 < \ldots < Y_5$ be the order statistics of a random sample of size $n = 5$ from a distribution with pdf $f(x;\theta) = e^{-|x-\theta|}/2$, $-\infty < x < \infty$, with θ unknown. Find the likelihood ratio test λ for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

9. Let $X_1, X_2, \ldots, X_n$ be iid from a distribution with CDF $F(x) = 1 - 1/x$ for $1 \leq x < \infty$, and zero otherwise.

   (a) Derive the CDF of the smallest order statistic $X_{(1)}$.

   (b) Find the limiting distribution of $X_{(1)}$.

   (c) Find the limiting distribution of $X_{(1)}^n$.

10. Let $X_1, \ldots, X_n$ be iid Uniform$(0, \theta)$ with $\theta > 0$, and $X_{(n)}$ be the largest order statistic.

    (a) Find the probability that the random interval $(X_{(n)}, 2X_{(n)})$ contains θ.

    (b) Find the constant c such that $(X_{(n)}, cX_{(n)})$ is a $100(1-\alpha)\%$ confidence interval for θ.

11. Let $X_1, \ldots, X_n$ be a random sample from the N$(\theta, 1)$ distribution, θ unknown. We wish to test $H_0 : \theta = \theta_0$ versus $H_A : \theta < \theta_0$. Find a test using the generalized likelihood ratio λ.

12. The distance in feet by which a parachutist misses a target is $D = \sqrt{X_1^2 + X_2^2}$, where X_1 and X_2 are independent with $X_i \sim N(0, 25)$. Find $\mathbb{P}(D \leq 12.25)$ (you can write down an integral for it, you don't need to evaluate it).