Ph.D. Qualification Examination in Statistics
September 2006

Correct and complete solutions to 5 problems guarantees a "pass."

1. Let $X_1, X_2, \ldots$ be an i.i.d. sample from the uniform distribution on $[\theta\,, 1+\theta]$, where $\theta > 0$ is unknown. Is there an unique maximum likelihood estimator of θ that is based on $\{X_i\}_{i=1}^{\infty}$? Prove or disprove.

2. Suppose $X_1, \ldots, X_n$ is an i.i.d. sample from the exponential density with mean θ, where $\theta > 0$ is unknown. Find an estimator of θ based on $X_1, \ldots, X_n$. Is your estimator complete? Is it sufficient? Justify your reasoning.

3. Let X have the binomial distribution with parameters $n = 10$ and $p = \theta$ unknown. Suppose we impose a uniform-$(0\,,1)$ prior on θ. Compute the posterior mean of θ. Is it consistent?

4. Let $X_1, X_2, \ldots$ denote an i.i.d. sample with common distribution function F. Define $F_n(x) = n^{-1}\sum_{j=1}^{n} \mathbf{I}\{X_j \le x\}$ for all $x \in (-\infty\,,\infty)$; i.e., F_n is the empirical distribution function based on the X_i's. Prove that for all $x \in (-\infty\,,\infty)$ fixed,

$$\mathrm{Var}\left(F_n(x) - F(x)\right) \le \frac{1}{4n}.$$

Use this to conclude that $F_n(x)$ is a consistent estimator for $F(x)$.

From here on let $X_1, X_2, \ldots$ denote an i.i.d. sample from an exponential density with mean $1/\theta$.

5. Is the MLE a consistent estimator of θ? Prove or disprove.

6. Find a confidence interval for θ that has asymptotic level $(1-\alpha)\%$.

7. Develop carefully the likelihood ratio test for $H_0 : \theta = 1$ versus $H_1 : \theta = 2$. Discuss the various optimality properties—if any—of your test.

8. Compute the distributions of:
(a) $\max_{1 \le i \le n} X_i$
(b) $\min_{1 \le i \le n} X_i$
(c) $X_1 + \cdots + X_n$.

Qualifying Exam in Statistics, 2005

Note: $I\{A\}$ denotes the indicator of the event A. You need the correct and complete solutions to at least 8 of these problems to pass.

1. Let $X_1, .., X_n$ be independent, identically distributed random variables with density function
$$f(t,\theta_1) = \begin{cases} 0, & \text{if} \quad t \notin [1,\infty) \\ \theta_1 t^{-\theta_1 - 1}, & \text{if} \quad 1 \le t < \infty \end{cases}$$
Let $Y_1, .., Y_m$ be independent, identically distributed random variables with density function
$$f(t,\theta_2) = \begin{cases} 0, & \text{if} \quad t \notin [1,\infty) \\ \theta_2 t^{-\theta_2 - 1}, & \text{if} \quad 1 \le t < \infty \end{cases}$$
We wish to test $H_0: \ \theta_1 = \theta_2$ against the alternative that $\theta_1 \neq \theta_2$. The two samples are independent.
(a) Find a test using the likelihood ratio.
(b) Provide a large sample approximation for the likelihood ratio test.

2. Let $X_1, .., X_n$ be independent random variables with density functions
$$f(t,\theta_i) = \begin{cases} 0, & \text{if} \quad t \notin [0,\infty) \\ \frac{1}{\theta_i} e^{-t/\theta_i}, & \text{if} \quad 0 \le t < \infty \end{cases}$$
We wish to test $H_0: \ \theta_1 = \theta_2 = \ldots = \theta_n$ against the alternative that H_0 is not true.
(a) Find a test using the likelihood ratio.
(b) Provide a large sample approximation for the likelihood ratio test.

3. Let $X_1, ..., X_n$ be independent, identically distributed random variables with density function
$$g(t;\theta) = \begin{cases} 0, & \text{if} \quad -\infty < t < \theta \\ e^{-(t-\theta)}, & \text{if} \quad \theta \le t < \infty. \end{cases}$$
We wish to test $H_0: \ \theta \le \theta_0$ against $H_A: \ \theta > \theta_0$.
(a) Find the uniformly most powerful test of size α.
(b) Compute the power function.

4. Let $X_1, ..., X_n$ be independent, identically distributed random variables with a continuous distribution function F. Let $x_{1/4}$ defined by $F(x_{1/4}) = 1/4$.
(a) Find a $1-\alpha$ lower confidence bound for $x_{1/4}$.
(b) Provide a large sample approximation.

5. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with distribution function

$$F(t) = \frac{1}{1+e^{-t}}, \quad -\infty < t < \infty.$$

Show that

$$Y_n = X_{n,n} - \log n$$

converges in distribution, where $X_{n,n} = \max\{X_1, X_2, \ldots, X_n\}$.

6. Let $X_1, ..., X_n$ be independent, identically distributed random variables with probability mass function

$$f(t, \theta) = \theta(1-\theta)^{t-1} I\{t = 1, 2, \ldots\} \;\; (0 < \theta < 1).$$

Find the uniformly minimum variance unbiased estimator for θ.

7. Let X and Y be independent random variables with densities

$$g(t) = \begin{cases} 0, & \text{if} \quad t \notin [-2, 2] \\ 1/4, & \text{if} \quad t \in [-2, 2] \end{cases}$$

and

$$f(t) = \begin{cases} 0, & \text{if} \quad t \notin [-1, 1] \\ 1/2, & \text{if} \quad t \in [-1, 1] \end{cases}$$

Compute the density of $X + Y$.

8. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with probability mass function

$$P\{X_i = 1\} = \theta \text{ and } P\{X_i = 0\} = 1 - \theta.$$

Let

$$\bar{X}_n = \frac{1}{n} \sum_{1 \le i \le n} X_i$$

and $h(t) = t^3 - t$. Find the sequences a_n and b_n such that $a_n(h(\bar{X}_n) - h(\theta) - b_n)$ has a non–degenerate limit distribution.

9. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with a continous probability distribution function F and $Y_1, Y_2, \ldots, Y_m$ be independent, identically distributed random variables with a continous probability distribution function H. The two samples are independent and $F = H$. Let

$$T = \sum_{1 \le i \le n} \sum_{1 \le j \le m} I\{X_i \le Y_j\}.$$

Compute ET and var$T$.

10. Let $\epsilon_i, 1 \leq i \leq n$ be independent identically distributed normal $N(0, \sigma^2)$ random variables and consider the model

$$y_i = i\alpha + \gamma i^2 + \epsilon_i \quad 1 \leq i \leq n.$$

We observe $y_i, 1 \leq i \leq n$.
(a) Find the maximum likelihood estimators for α, γ and σ^2
(b) Compute the joint distribution of the maximum likelihood estimators for (α, γ).

2004 Qualifying Exam in Statistics

1. Let $U_1, U_2, \dots U_n$ be independent identically distributed random variables, uniformly distributed on $[0,1]$. Compute the covariance between $U_{1,n} = \min_{1\le i\le n} U_i$ and $U_{n,n} = \max_{1\le i\le n} U_i$.

2. Let $X_1, X_2, \dots X_n$ be independent, identically distributed normal $N(\mu, \sigma^2)$ random variables. We wish to test $H_0 : \mu = \mu_0$ agains $H_A : \mu \neq \mu_0$ (σ is unknown). Show that the t–test and the likelihood tests are equivalent.

3. Let X_1 and X_2 be two independnet, identically distributed exponential (1) random variables. Compute the joint density of $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$.

4. Let $X_1, X_2, \dots X_n$ be independent, identically distributed uniform on $[0, \theta], \theta > 0$ random variables. We wish to test $H_0 : \theta \le \theta_0$ against $H_A : \theta > \theta_0$. Find the unifromly most powerful test of size α and compute the power function. (You must explain why your test has this property.)

5. Let $X_1, X_2, \dots X_n$ be independent, identically distributed random variables with density function

$$f(t,\theta) = \begin{cases} 0, & \text{if } t \notin [0,\theta] \\ \frac{1}{\theta}, & \text{if } t \in [0,\theta]. \end{cases}$$

   We use $\hat{\theta} = cX_{1,n}$ and $\hat{\eta} = \alpha\bar{X}$ to estimate θ, where

$$X_{1,n} = \min_{1\le i\le n} X_i \text{ and } \bar{X} = \frac{1}{n}\sum_{1\le i\le n} X_i.$$

   (a) Find c and α such that the estimators are unbiased.
   (b) Compute the relative efficiency.

6. Let $X_1, X_2, \dots X_n$ be independent, identically distributed Poisson (θ) random variables. Find the uniformly minimum unbiased estimator for $\eta = \theta^2 e^{-\theta}$. (You must explain your answer.)

7. Let $X_1, X_2, \dots X_n$ and $Y_1, Y_2, \dots Y_m$ be independent, identically distributed random variables with a common continous distribution function. Compute the expected value and the variance of

$$Z = \sum_{1\le i\le n} \sum_{1\le j\le n} I\{X_i \le Y_j\},$$

where $I\{A\}$ denotes the indicator function of the event A.

8. Let $X_1, X_2, \ldots X_n$ be independent, identically distributed random variables with density function

$$f(t,\theta) = \begin{cases} 0, & \text{if } t \le \theta \\ e^{-(t-\theta)}, & \text{if } t > \theta. \end{cases}$$

(a) Find two $1-\alpha$ equal–tail confidence intervals for θ. One should be based on the smallest order statistic and the other one should be based on the sample mean.
(b) Compare the expected lengths of the confidence intervals.

9. Let ξ_n and η_n be two sequences of random variables. Prove or give counterexamples to the following statemenets:
(a) If $\xi_n \xrightarrow{d} \xi$ and $\eta_n \xrightarrow{d} \eta$, then $\xi_n + \eta_n \xrightarrow{d} \xi + \eta$.
(b) $\xi_n \xrightarrow{P} \xi$ and $\eta_n \xrightarrow{P} \eta$, then $\xi_n + \eta_n \xrightarrow{P} \xi + \eta$.
(c) $(\xi_n, \eta_n) \xrightarrow{P} (\xi, \eta)$ then $\xi_n + \eta_n \xrightarrow{P} \xi + \eta$.

($\xrightarrow{d}$ and $\xrightarrow{P}$ denote convergence in distribution and convergence in probability.)

10. Let $X_1, X_2, \ldots X_n$ be independent, identically distributed random variables with $P\{X_i = 0\} = \theta$ and $P\{X_i = 1\} = 1 - \theta$ $(0 < \theta < 1)$. Find the numerical sequences a_n and b_n such that

$$Y_n = a_n \hat{X}_n (1 - \hat{X}_n) - b_n$$

converges in distribution to a nondegenerate random variables, where

$$\hat{X}_n = \frac{1}{n} \sum_{1 \le i \le n} X_i.$$

STATISTICS

PhD Written Qualifying Examination – 15 Aug. 2000 9:00–12:00

Rules:

1. No books nor notes can be used.
2. No calculators nor tables can be used.
3. The value of each question is 10 points. You need at least 75 points to pass.
4. You must explain all your answers. You must state all theorems you used to solve the questions.

1. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed exponential (1) random variabes with common density function

$$f(t) = \begin{cases} 0, & \text{if } -\infty < t < 0 \\ e^{-t}, & \text{if } 0 \leq t < \infty. \end{cases}$$

(a) Show that $(\sum_{1 \leq i \leq n} X_i)^{1/2} - n^{1/2}$ converges in distribution to a normal random variable.
(b) Compute the mean and the variance of the limiting normal random variable.

2. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed Poisson random variables with common probability mass function

$$p(k, \theta) = \frac{\theta^k}{k!} e^{-\theta} \quad k = 0, 1, \ldots$$

Let

$$\tau = \theta e^{-\theta}.$$

Find the uniformly minimum variance unbiased estimator for τ.

3. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed uniform (0,θ) random variables with common density function

$$f(t) = \begin{cases} 0, & \text{if } t \notin [0,\theta] \\ 1/\theta, & \text{if } t \in [0,\theta]. \end{cases}$$

(a) Find the maximum likelihood estimator for θ.
(b) Find a $1-\alpha$ equal–tail confidence interval for θ.

4. Let $X_1, X_2, \ldots, X_n$ be independent exponential (θ_i) random variables, i.e. the density of X_i is

$$f(t,\theta_i) = \begin{cases} 0, & \text{if } -\infty < t < 0 \\ \frac{1}{\theta_i} e^{-t/\theta_i}, & \text{if } 0 \le t < \infty. \end{cases}$$

We wish to test $H_0: \quad \theta_1 = \theta_2 = \ldots = \theta_n$ aginst the alternative that H_0 is not true.
(a) Find the likelihood ratio test.
(b) Show that the likelihood ratio test is distribution free under H_0.
(c) Show that the likelihood ratio test is equivalent with the χ^2 test under H_0.
(d) Approximate the critical values of the likelihood ratio test under H_0.

5. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with common density function f. Let

$$\hat{f}_n(t) = \frac{1}{nh} \sum_{1 \le i \le n} K\left(\frac{t - X_i}{h}\right),$$

where K is a symmetric density function and $K(t) = 0$ if $|t| > 1$.. We assume that $h = h(n) \to 0$ as $n \to \infty$. We assume that the second derivative of f exists and continuous in a neighbourhood of t. We use $\hat{f}$ to estimate f. Compute the bias of $\hat{f}_n(t)$.

6. X and Y are two independent identically distributed normal $N(0, \sigma^2)$ random variables. Let

$$Z = X^2 + Y^2$$

and

$$W = \frac{X}{(X^2 + Y^2)^{1/2}}.$$

(a) Show that Z and W are independent.
(b) Compute the marginal distributions of Z and W.

7. Z_1 and Z_2 are two independent identically distributed exponential (λ) random variables, i.e. the common density is

$$f(t) = \begin{cases} 0, & \text{if } -\infty < t < 0 \\ \frac{1}{\lambda} e^{-t/\lambda}, & \text{if } 0 \leq t < \infty. \end{cases}$$

Let $X = Z_1$ and $Y = Z_1 + Z_1 Z_2$. Compute
(a) $E(Y|X = x)$
(b) $\text{var}(E(Y|X))$.

8. Let $y_i = \alpha x_i + \epsilon_i$ where ϵ_i, $1 \leq i \leq n$ are independent identically distributed normal $N(0, \sigma^2)$ random variables.
(a) Find the maximum likelihood estimators for α and σ^2.
(b) Find the distributions of the maximum likelihood estimators.
(c) Compute the least–squares estimator for α.

9. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed Poisson random variables with common probability mass function

$$p(k, \theta) = \frac{\theta^k}{k!} e^{-\theta} \quad k = 0, 1, \ldots$$

Using the Cramér–Rao inequality show that the sample mean is a uniformly minimum variance unbiased estimator for θ.

10. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed exponential (λ) random variables. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed exponential (μ) random variables. The two samples are independent. We wish to test $H_0 : \theta = \mu$ agians the alternative that H_0 is not true.
(a) Find the likelihood ratio test.
(b) Find the asymptotic distribution of the likelihood ratio under H_0.
(c) Find an equal–tail confidence interval for $\tau = \lambda/\mu$.

Exam in Statistics 20 Aug 1999

Rules:

1. You have 3 hours to work on the problems.
2. No books nor notes can be used. You can use a calculator.
3. You need 100 points to pass.

1. Let $X_1, X_2, \ldots, X_n$ be independent random variables. We assume that X_i has a Poisson(λ_i) distribution. We wish to test

$$H_0: \ \lambda_1 = \lambda_2 = \ldots = \lambda_n$$

against the alternative

$$H_A: \ H_0 \text{ is not true.}$$

1. Show that the likelihood ratio test and the χ^2 tests are equivalent under H_0. (15 points)
2. Explain why and how the normal as well as the χ^2 distributions can ve used to approximate the critical values. (5 points)

2. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with a continuous distribution function F. The ordered observations are $X_{1,n} \leq X_{2,n} \leq \ldots X_{n,n}$. Let

$$T_n = \sum_{i=1}^{n-1} (F(X_{i+1,n}) - F(X_{i,n}))^2.$$

Show that there are numerical sequences a_n and b_n such that $(T_n - a_n)/b_n$ converges in distribution to a standard normal random variable. (20 points)

3. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed normal $\mathrm{N}(0, \sigma^2)$ random variables. We wish to test

$$H_0: \ \sigma \leq \sigma_0$$

against

$$H_A: \ \sigma > \sigma_0.$$

1. Find the uniformly most powerful test of size α. (10 points)
2. Compute the rejection region as a function of α. (10 points)

4. Consider the following model:

$$y_i = f(x_i) + \epsilon_i \quad 1 \leq i \leq n$$

where

$$f(t) = a_0 + a_1 t + \ldots + a_p t^p.$$

The parameters $a_0, a_1, \ldots, a_p$ are unkown. The errors $\epsilon_1, \epsilon_2 \ldots, \epsilon_n$ are independent normal $\mathrm{N}(0, \sigma^2)$ random variables. It is assumed that $x_i \in [0, 1]$ and known. We observe (y_i, x_i) $1 \leq i \leq n$.
1. Estimate the parameters $a_0, a_1, \ldots, a_p$ and σ^2. (15 points)
2. Get the distribution of the estimator for σ^2. (5 points)

5. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with distribution function F. We assume that $f = F'$ has a bounded second derivative. $K(t)$ is a non–negative function satisfying $K(t) = K(-t)$ (symmetric),

$$\int_{-\infty}^{\infty} K(t)dt = 1 \text{ and } K(t) = 0 \text{ if } t \notin [-1, 1].$$

The sequence $h = h(n)$ is positive and goes to 0. We estimate f with

$$\hat{f}_n = \frac{1}{nh} \sum_{1 \leq i \leq n} K(\frac{t - X_i}{h}).$$

1. Compute B_n, the bias of $\hat{f}_n$. (10 points)
2. Compute the limit of $B_n/h^2(n)$ as $n \to \infty$. (10 points)

6. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables uniform on $[\theta, \theta + 1]$.
1. Find a sufficient statistic for θ. (5 points)
2. Is your statistic complete? (5 points)
3. Find the maximum likelihood estimator for θ. (10 points)

7. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be independent random samples of independent observations. The distribution of X_i is normal $\mathrm{N}(\mu_1, \sigma_1^2)$ and the distribution of Y_i is normal $\mathrm{N}(\mu_2, \sigma_2^2)$. The values of σ_1 and σ_2 are known. We wish to test $H_0: \quad \mu_1 = \mu_2$ against $H_A: \quad \mu_1 \neq \mu_2$. Show that the likelihood ratio test and the t–test are equivalent under H_0. (20 points)

8. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed normal $\mathrm{N}(\mu, \sigma^2)$ random variables. Let

$$U_n = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} I\{|X_i - X_j| > 1\}.$$

1. Compute EU_n. (10 points)
2. Assume that σ is known. We wish to test $H_0 : \ \mu = \mu_0$ (μ_0 is given) against the alternative that H_0 is not true. Can we use U_n to test H_0 against this alternative? (5 points)
3. Now μ and also σ are unknown. We wish to test the null–hypothesis that $\sigma = \sigma_0$ against the alternative that H_0 is not true. Can we use U_n now? (5 points)

Preliminary Examination 1998: Probability & Statistics

Instructions: You pass this exam if all the following conditions are satisfied.

(i) You got at least 15 points from probability

(ii) You got at least 15 points from statistics

(iii) You got at least 45 points total

Probability

1. Suppose $\{E_1, E_2, \cdots\}$ is a sequence of measurable (otherwise arbitrary) events. Suppose $\{F_1, F_2, \cdots\}$ is another sequence of measurable events also totally independent of all of the E_i's. Assume the following:

(a) $P(E_k,$ infinitely often $) = 1$;

(b) there exists some $p > 0$, such that for all $k \geq 1$, $P(F_k) \geq p$.

Prove: $P(E_k \cap F_k,$ infinitely often $) \geq p > 0$. (10 points)

2. Let $X_1, X_2, \ldots$ be independent identically distributed positive random variables with $EX_1 = \mu$ and $\text{var}(X_1) = \sigma^2$. Let

$$N(t) = \min\{k : \sum_{1 \leq i \leq k} X_i > t\}, \quad 0 < t < \infty.$$

(a) Show that

$$\lim_{t \to \infty} \frac{1}{t} N(t) = \frac{1}{\mu}$$

almost surely. (5 points)

(b) Show that $(N(t) - t/\mu)/(t\sigma^2/\mu^3)^{1/2}$ goes in distribution to a standard normal random variable. (5 points)

3. Let $X_1, X_2, \ldots$ be independent random variables with distribution functions $F_1, F_2, F_3, \ldots$. Let $Y = \sup_{1 \leq i < \infty} X_i$.

(a) Show that $P\{Y < \infty\}$ is 0 or 1 depending on whether $\sum_{1 \leq i < \infty}(1 - F_i(x))$ converges for some x. (5 points)

(b) Show that if $P\{Y < \infty\} = 1$, then $\prod_{1 \leq i < \infty} F_i(x)$ converges for all x and it is the distribution function of Y.

4. Let X_1, X_2, X_3 be i.i.d., each with an exponential distribution with mean 1. Find the joint distribution of

$$Y_1 = \frac{X_1}{X_1 + X_2}, \qquad Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}, \qquad Y_3 = X_1 + X_2 + X_3.$$

(7 points) In particular, are they independent? (3 points)

5. Suppose N is a positive random variable.

(a) Show that

$$P(N > 0) \geq \frac{(EN)^2}{EN^2}.$$

(5 points)

(b) Let $X_1, X_2, \cdots$ be i.i.d. random variables. Define $S_n = X_1 + \cdots + X_n$. Use part (a) to show that for any positive integer n,

$$P(S_j = 0, \text{ for some } 1 \le j \le n) \ge \frac{\sum_{j=1}^{n} P(S_j = 0)}{2\left(1 + \sum_{j=1}^{n} P(S_j = 0)\right)}.$$

(5 points)

Statistics

6.Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with distribution function F and density function f. We assume that the second derivative of f is bounded. We estimate f by

$$f_n(t) = \frac{F_n(t + h_n) - F_n(t - h_n)}{2h_n},$$

where F_n is the empirical distribution function and $h_n \to 0$ as $n \to \infty$.

(a) Show that $Ef_n(t) \to f(t)$ as $n \to \infty$. (5 points)

(b) Show that if $nh_n \to \infty$, then

$$\frac{f_n(t) - Ef_n(t)}{(var(f_n(t)))^{1/2}}$$

is asymptotically standard normal as $n \to \infty$. (5 points)

7. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a Poisson distribution with parameter $\theta > 0$.

(a) Show that $(-1)^{X_1}$ is unbiased for $e^{-2\theta}$, and use the Lehmann–Scheffé theorem to deduce a UMVUE of $e^{-2\theta}$ based on the sample. (Hint: The conditional distribution of X_1 given $X_1 + \cdots + X_n$ has a simple form.) (5 points)

(b) Argue that the UMVUE in part (a) is a consistent estimator of $e^{-2\theta}$. (5 points)

8. Let $X_1, \ldots, X_n$ be a random sample from the $N(\mu, \sigma^2)$ distribution, with both μ and σ^2 unknown.

(a) Derive the likelihood ratio test for $H : \sigma^2 = \sigma_0^2$ against all alternatives. Here σ_0^2 is a known positive constant. (5 points)

(b) Do the same for $H : \sigma^2 \ge \sigma_0^2$. (5 points)

9. Let $X_1, \ldots, X_n$ be a random sample from $N(\theta, \theta)$ (i.e., normal with mean and variance both equal to θ), where $\theta > 0$.

(a) Give three pivotal quantities (or pivots) involving the entire sample, which respectively have a standard normal distribution, a t-distribution, and a chi-squared distribution, and indicate the numbers of degrees of freedom. (5 points)

(b) Use the normal pivotal quantity in part (a) to obtain a $100(1 - \alpha)$ percent confidence interval for θ. (5 points)

10. Let $X_1, X_2, \ldots, X_n$ be independent Poisson random variables with parameters $\theta_1, \theta_2, \ldots, \theta_n$. We wish to test $H_o: \theta_1 = \theta_2 = \ldots = \theta_n$ against the alternative that H_o is not true.

(a) Find the likelihood ratio test. (5 points)

(b) Show that the likelihood ratio is asymptotically normal (after centralizing and normalizing) under H_o. (5 points)

Preliminary Examination 1996: Probability & Statistics

Instructions: Choose 6 of the 10 problems, with at least two from probability and at least two from statistics, and write up solutions for these six problems *only*. 70 percent correct will be a passing score.

Probability

In Problems 1 and 2, $S_n = X_1 + \cdots + X_n$.

1. Prove Cantelli's theorem: If $X_1, X_2, \ldots$ are independent (but not necessarily identically distributed), mean zero, random variables with $\sup_n E[X_n^4] < \infty$, then $S_n/n \to 0$ a.s. as $n \to \infty$.

2. (a) Let $X_1, X_2, \ldots$ be i.i.d. Poisson random variables with mean 1, and let Z be $N(0,1)$. Denote $a^- = -\min\{a, 0\}$, and prove that

$$(*) \qquad E\left[\left(\frac{S_n - n}{\sqrt{n}}\right)^-\right] \to E[Z^-].$$

(b) Evaluate both expectations in (*) explicitly, and, noting the telescoping sum, deduce Stirling's formula for $n!$.

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, and let M be the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ consisting of the $\mathcal{G}$-measurable functions in L^2. Show that $T(X) = E[X \mid \mathcal{G}]$ coincides with the orthogonal projection of $L^2(\Omega, \mathcal{F}, P)$ onto M.

4. Consider the Markov chain in the state space $S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ with transitions $P(i, i+2) = p$ and $P(i, i-1) = 1 - p$, where $0 < p < 1$. Determine for which p this chain is recurrent and for which p it is transient.

5. Let $\{B(t),\ t \geq 0\}$ be a standard Brownian motion. Prove directly, using

$$V_n = \sum_{i=1}^{2^n} |B(i/2^n) - B((i-1)/2^n)|,$$

that $B(\cdot, \omega)$ is of unbounded variation on [0,1] for a.e. ω.

Statistics

6. Suppose $X_1, \ldots, X_n$ is an i.i.d. sample from a normal population with

$$EX_1 = \mathrm{Var}(X_1) = \mu > 0.$$

(a) Compute the maximum likelihood estimator $\hat{\mu}$ of μ;
(b) Is $\hat{\mu}$ consistent?
(c) Is $\hat{\mu}$ asymptotically normal?

7. Suppose $X_1, \ldots, X_n$ is an i.i.d. sample from a normal distribution with mean μ and variance 1.
(a) Find the UMVU estimator for μ. (Prove the optimality criterion.)
(b) Put a $N(\theta, \tau^2)$ prior on μ and find the minimax estimator of μ.

8. Consider the linear model: $Y_i = \beta + \epsilon_i$, $1 \leq i \leq n$. Here, ϵ_i's are i.i.d. $N(0, \sigma^2)$, and σ and β are unknown.
(a) Find the least squares estimator of β;
(b) Find the UMVU estimator for β (prove the optimality);
(c) Find the UMP test for $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_1$;
(d) Discuss—without proofs—how to find the UMP test for $H_0 : \beta > \beta_0$ vs. $H_1 : \beta \leq \beta_0$ from tests of the form in part (c) above.

9. Let $X_1, \ldots, X_n$ be an i.i.d. sample which is uniformly chosen from the interval (θ_1, θ_2).
(a) Prove that there are no one-dimensional (i.e., not vector-valued) sufficient statistics for $h(\theta_1, \theta_2)$ where h is a one-to-one measurable function.
(b) Is there a one-dimensional (i.e., not vector-valued) sufficient statistics for $\mu = (\theta_1 + \theta_2)/2$?

10. Let $X_1, \ldots, X_n$ be an i.i.d. sample from a uniform $(0, \theta)$ distribution, where $\theta > 0$.
(a) Find the maximum likelihood estimator $\hat{\theta}$ for θ;
(b) Construct a $100(1-\alpha)\%$ confidence interval for θ based on $\hat{\theta}$.

Preliminary Examination

PROBABILITY & STATISTICS

1994

You have 2 hours to complete this test.

Answer as many questions as you can. In order to insure a pass, you will need to solve as many as five questions total, with 1 complete solution in each subject.

This is an open book examination.

Probability Questions

1. Let $X_1, X_2, \cdots$ be independent, identically distributed random variables, uniformly distributed on $[0,1]$. Show that

$$Y_n = \frac{4\sum_{1\le k\le n} kX_k - n^2}{n^{3/2}},$$

converges in distribution to a normal random variable.

2. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\{B(t), 0 \le t \le 1\}$ be a Brownian motion on it. Since almost all sample paths of B are continuous, $\int_0^t B(t)dt$ can be defined as a usual Rieman integral. Compute the distribution of $\int_0^1 B(t)dt$.

3. Let $\{S_n, \mathcal{F}_n,\ n \ge 1\}$ be a nonnegative martingale with $ES_n = 1$. Show that for all $\lambda > 0$,

$$P\{S_n \ge \lambda, \text{ for some } n \ge 1\} \le \frac{1}{\lambda}.$$

4. Let $X_1, X_2, \cdots$ be independent, identically distributed random variables. Show that the following statements are equivalent:

(a) $E|X_1|^\nu < \infty$;

(b) $X_n/n^{1/\nu} \to 0$, almost surely;

(c) $\max_{1\le i\le n} |X_i|/n^{1/\nu} \to 0$, almost surely.

5. Let $X_1, X_2, \cdots$ be independent, identically distributed normal random variables. Find two numerical sequences, a_n and b_n, such that

$$\frac{\max_{1\le i\le n} X_i - a_n}{b_n}$$

converges in distribution to a non–degenerate random variable.

6. Let $0 \le X_n \le 1$ be adapted to $\mathcal{F}_n$. Let $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and suppose

$$P(X_{n+1} = \alpha + \beta X_n \mid \mathcal{F}_n) = X_n$$
$$P(X_{n+1} = \beta X_n \mid \mathcal{F}_n) = 1 - X_n.$$

(a) Show that $P\{\lim_{n\to\infty} X_n = 0 \text{ or } 1\} = 1$.

(b) Show that if $X_0 = \theta$, then $P\{\lim_{n\to\infty} X_n = 1\} = \theta$.

1. Suppose $X_1, \cdots X_m$ are independent with $X_j \sim \text{BIN}(n_j, p_j)$.

(a) Find the UMVUE's of $p_1, \cdots p_m$.

(b) Suppose you know that $p_1 = p_2 = \cdots = p_m$. Let p denote this common (but unknown) value. Find the UMVUE of p.

(c) Find the likelihood ratio statistic, λ, for $H_0 : p_1, = \cdots = p_m$ versus $H_1 : p_i \neq p_j$, for some i and j.

(d) It can be shown that $2 \log \lambda$ is approximately the same as the usual χ^2 statistics. Using this fact, find an approximate test for H_0 vs H_1 above.

2. Consider the linear model:

$$Y_{ij} = \beta_i + \varepsilon_{ij}, \qquad 1 \leq i \leq 2, \qquad 1 \leq j \leq J.$$

Suppose ε_{ij}'s are independent and for some (known) a_1 and a_2, $\varepsilon_{ij} \sim \text{N}(0, a_i^2 \sigma^2)$.

(a) Find the U.M.V.U.E.'s of β_1 and β_2.

(b) Suppose you know that for some unknown β, $\beta_1 = \beta_2$. Find the U.M.V.U.E. of β.

3. Suppose $X \sim \text{POISS}(\theta)$. Put a $\text{GAMMA}(\alpha, \beta)$ prior on θ and suppose we have the following loss function: $\ell(\theta, a) = (a - \theta)^2/\theta$. Find the Bayes' estimator of θ.

4. Suppose $\theta \in \{\theta_0, \theta_1\}$ is an unknown (θ_0 and θ_1 are, however, known.) Put some prior, π, on θ. We are to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$. Our actions are 0 (accept) and 1 (reject). Find the Bayes' procedure for doing this test, if the loss function, $\ell(\theta, a)$, is the 0–1 loss given by:

$$\ell(0,1) = 0, \qquad \ell(0,0) = 1$$
$$\ell(1,0) = 0, \qquad \ell(1,1) = 1.$$

Is this procedure minimax?

5. Construct a $(1 - \alpha)$ two–sided confidence interval for the correlation coefficient of a bivariate normal distribution. (HINT. This is an exponential family.)

Some Densities

$\mathrm{N}(\mu, \sigma^2)$ $\qquad x \in \mathbb{R}^1$: $\quad \dfrac{1}{\sqrt{2\pi}\sigma} \exp\big(-\dfrac{1}{2\sigma^2}(x-\mu)^2\big).$

$\mathrm{Gamma}(\alpha, \beta)$ $\qquad x \in \mathbb{R}^1$: $\quad \dfrac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\int_0^\infty t^{\alpha-1} e^{-t} dt}.$

$\mathrm{Poiss}(\lambda)$ $\qquad x = 0, 1, \cdots$: $\quad \dfrac{1}{x!} e^{-\lambda} \lambda^x.$

Bivariate Normal$(\mu_1, \mu_2, \rho, \sigma_1, \sigma_2)$ $\qquad (x, y) \in \mathbb{R}^2$:

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right\}\right).$$

1993 Prelim in Probability and Statistics

There are ten problems. Each counts 10 points. The minimum passing score is 60 points.

1. Let $X_0 = 1$ and define X_n inductively by declaring that X_{n+1} is uniformly distributed over $(0, X_n)$. Prove that $n^{-1} \log X_n \to c$ a.s. and compute c.

2. Let $X_1, X_2, \ldots$ be independent Poisson random variables with $EX_n = \lambda_n$ and let $S_n = X_1 + \cdots + X_n$. Show that if $\sum \lambda_n = \infty$, then $S_n / ES_n \to 1$ a.s.

   Hint: Show that (*) $Y_n / c_n \to 1$ a.s., provided that $Y_n \geq 0$ is nondecreasing in n, and (*) holds for a subsequence $n(k)$ that has $c_{n(k+1)}/c_{n(k)} \to 1$.

3. Let $X_1, X_2, \ldots$ be i.i.d. mean 0, variance $\sigma^2 \in (0, \infty)$.

   (a) Use the central limit theorem as well as Kolmogorov's 0–1 law to conclude that $\limsup S_n/\sqrt{n} = \infty$ a.s.

   (b) Show that $S_n/\sqrt{n}$ does not converge in probability.

4. Suppose that X and Y are independent. Let f be o Borel function on $\mathbf{R}^2$ with $E[|f(X,Y)|] < \infty$ and let $g(x) = E[f(x,Y)]$. Show that $E[f(X,Y) \mid X] = g(X)$.

5. A thinker who owns r umbrellas travels back and forth between home and office, taking along an umbrella (if there is one at hand) in rain (probability p) but not otherwise (probability $q = 1 - p$). Let the state be the number of umbrellas at hand, irrespective of whether the thinker is at home or at work. Set up the transition matrix, and show that the Markov chain approaches equilibrium (i.e., the ergodic theorem is applicable). Find the steady-state probability of his getting wet, and show that five umbrellas will protect him at the 5% level against any climate (any p).

6. Let $U_1, U_2, \ldots, U_n$ be independent identically distributed random variables, uniform on [0,1]. Let $U_{1,n} \le U_{2,n} \le \ldots \le U_{n,n}$ denote the order statistics. Let $X_1, X_2, \ldots, X_{n+1}$ be independent identically distributed exponential random variables with $EX_1 = 1$. Define $S(i) = X_1 + \cdots + X_i$.

   (a) Prove that the random vectors $\{U_{1,n}, \ldots, U_{n,n}\}$ and $\left\{\frac{S(1)}{S(n+1)}, \ldots, \frac{S(n)}{S(n+1)}\right\}$ have the same distribution.

   (b) Compute the asymptotic distribution of $n(U_{i+3,n} - U_{i,n})$, as $n \to \infty$ when i is fixed.

7. Let $Y_i = \alpha\frac{i}{n} + \varepsilon_i, \quad 1 \le i \le n$. We assume that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are independent, identically distributed random variables with $E\varepsilon_i = 0$, $0 < \sigma^2 = var\ \varepsilon_i < \infty$ and $E\varepsilon_i^4 < \infty$.

   (a) Find the least-squares estimator for α.

   (b) Show that the estimator is asymptotically normal.

   (c) Find an estimator for σ^2.

8. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables, uniformly distributed on $[0, \theta]$, $\theta > \theta_0$. We want to test $H_0 : \theta = \theta_0$ against $H_A : \theta > \theta_0$.

   (a) Find the uniformly most powerful test. (You must prove your claim.)

   (b) Show that the uniformly most powerful test and the likelihood ratio test are equivalent.

   (c) Compute the power function of the most powerful test.

9. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function f. We assume that f' is bounded. Let K be a function satisfying $\int_{-\infty}^{\infty} K(u)du = 1$, $K'(u)$ is bounded and $K(u) = 0$, if $|u| \ge a$ where a is a constant. The density f is estimated by

$$\hat{f}_n(t) = \frac{1}{nh} \sum_{1 \le i \le n} K\left(\frac{t - X_i}{h}\right).$$

Show that $\hat{f}_n(t)$ is an almost surely uniformly consistent estimator for f on $[\alpha, \beta]$, $-\infty < \alpha < \beta < \infty$.

10. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with $P\{X_i = 1\} = p, \quad P\{X_i = 0\} = 1 - p$.

    (a) Compute the maximum likelihood estimator of $\sigma^2 = p(1-p)$.

    (b) Compute the bias, the variance and the mean-square error of the estimator.

    (c) Is the estimator asymptotically efficient?

    (d) Find the uniformly minimum variance unbiased estimator for σ^2.