Statistics Prelim Exam
University of Utah
Department of Mathematics

August 2016

Read the following instructions before you begin:

- You may attempt all of 10 problems in this exam. However, you can turn in solutions for **at most** 6 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 5080–5090/6010 texts, then you need to carefully state that result.

Exam problems begin here:

1. Let $X_1, X_2, \ldots, X_n$ be iid with common pdf $f(x) = 2x, 0 < x < 1$ and zero elsewhere. Find the limiting distribution of $n(1 - X_{(n)})$, where $X_{(n)} = \max\{X_1, \ldots, X_n\}$.
2. Let $X_1 \sim N(0, 4\theta)$, $X_2 \sim N(0, 9\theta)$, $X_3 \sim N(0, 4\theta^2)$, $X_4 \sim N(0, 12\theta^2)$, and assume that all Xs are independent of each other. Using all four X variables and the t distribution, compute a $100(1-\alpha)\%$ confidence interval for θ.
3. Let $X_1, X_2, \ldots, X_n$ be iid with common pdf
$$f(x; \theta) = \frac{1}{2\theta} e^{-|x|/\theta}, \quad -\infty < x < \infty,$$
with θ unknown but positive. Compute the Cramer-Rao lower bound for the variance of an unbiased estimator of θ, and extend your calculation to finding the UMVUE by finding a linear combination of
$$\sum_{i=1}^{n} \frac{\partial}{\partial \theta} \log f(X_i; \theta)$$

that is unbiased.

4. If a random sample of size n is taken from a distribution having pdf $f(x;\theta) = 2x/\theta^2, 0 < x \leq \theta$ and zero elsewhere, find:

   (a) the MLE $\hat{\theta}$ for θ,

   (b) the constant c so that $\mathrm{E}[c\hat{\theta}] = \theta$,

   (c) the MLE for the median of the distribution.

5. If $X_1, X_2, \ldots, X_n$ is a random sample from a distribution having pdf of the form $f(x;\theta) = \theta x^{\theta-1}, 0 < x < 1$ and zero elsewhere, show that a best critical region for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$ is

$$C = \left\{ (x_1, x_2, \ldots, x_n) : c \leq \prod_{i=1}^{n} x_i \right\}.$$

6. Let $X_1, X_2, \ldots, X_{25}$ denote a random sample of size 25 from a normal distribution with mean θ and variance 100. Find a uniformly most powerful critical region of size $\alpha = 0.10$ for testing $H_0 : \theta = 75$ against $H_1 : \theta > 75$.

7. Let X_1 and X_2 be two independent random variables. Suppose that X_1 is $\chi^2(r_1)$ and $X_1 + X_2$ is $\chi^2(r)$, where r_1, r are the respective degrees of freedom and $r_1 < r$. Show that X_2 is $\chi^2(r - r_1)$.

8. Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables, each with the $N(\beta x_i, \gamma^2 x_i^2)$ distribution, where the numbers $x_1, x_2, \ldots, x_n$ are known, not all are equal, and none of them is zero. Suppose β and γ are unknown. Find the maximum likelihood estimators of β and γ^2.

9. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Gamma(α, θ) distribution, with α known and θ unknown. Find the Fisher information of this distribution and show that the MLE of θ is an efficient estimator of θ.

10. Suppose that an engineer wishes to compare the number of complaints per week filed by union stewards for two different shifts at a manufacturing plant. It is known that the number of complaints per week on the ith shift has a Poisson distribution with mean θ_i for $i = 1, 2$. One hundred independent observations on the number of complaints gave means $\bar{x} = 20$ for shift 1 and $\bar{y} = 22$ for shift 2. Using this data test $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$ by the likelihood ratio method, with $\alpha = 0.01$. (**Hint:** First find the likelihood ratio test statistic for λ and then use the distribution of $-2 \log \lambda$.)