UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Real Analysis
August 17, 2022.

**Instructions.** Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points. Carefully state any theorems you use.

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1. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to Y$ be a function.
   (a) Show that $\mathcal{N} = \{f^{-1}A : A \in \mathcal{M}\}$ is a $\sigma$-algebra.
   (b) Let $\nu(B) = \mu(f(B))$ for $B \in \mathcal{N}$. Show that $\nu$ is a measure.
   (c) Show that even if $(X, \mathcal{M}, \mu)$ is complete we do not necessarily have that $(Y, \mathcal{N}, \nu)$ is complete.
2. Define the weak topology on $L^2(\mathbb{R}, \lambda)$ and prove that it is not metrizable.
3. Let $f : [0, 1] \to [0, \infty]$ be measurable. Show that $U(f) = \{(x, y) : 0 \leq y \leq f(x)\})$ is measurable and $\int f d\lambda = \lambda^2(U(f))$.
4. State the Lebesgue dominated convergence theorem. Show that the converse is not true. That is, there is a sequence of functions that satisfy the conclusion of the Lebesgue dominated convergence theorem but not its assumption.
5. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ be its Borel $\sigma$-algebra. Show that if $(X, \mathcal{B}, \mu)$ is a $\sigma$-finite measure space that is outer regular on finite measure, measurable subsets, then it is automatically outer regular on all measurable subsets.

   Recall that $\mu$ is called outer regular on a collection of subsets $\mathcal{S}$ if for every $A \in \mathcal{S}$ and $\epsilon > 0$ there is an open set $U$ so that $A \subset U$ and $\mu(U \setminus A) < \epsilon$.
6. State the open mapping theorem for Banach spaces and use it to prove the following: Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $(\mathcal{C}, \|\cdot\|_{\mathcal{C}})$ be Banach spaces and $A : \mathcal{B} \to \mathcal{C}$ be continuous and linear. If $A$ is a bijection then $A^{-1}$ is continuous.