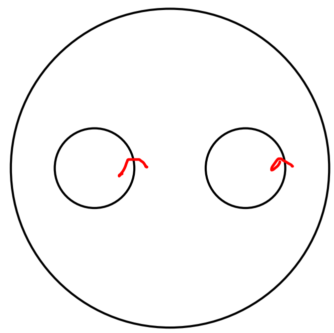
Math 6520 Qualifying Exam, August 2022

Answer at most five of the problems below. Each is worth 10 points. If you answer more than five problems, let me know which of the five you would like me to grade. For a high pass, you have to solve at least three problems completely. For a pass, you should solve two problems completely and get at least 25 points. Show all your work, and provide reasonable justification for your answers. ***Please state carefully any (major) results you use.***

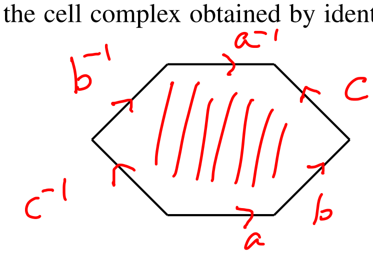
1. Let X be the compact surface obtained from the disk by deleting two disjoint open disks, as in the figure below. Fix orientations on two boundary components as in the figure. Identify the boundaries using each of the two possible orientations on the third component. Show that the two quotient spaces so obtained have non-isomorphic fundamental groups. Hint: abelianize.



2. Draw a picture of a covering space of $S^1 \vee S^1$ whose fundamental group is the normal subgroup of $\pi_1(S^1 \vee S^1, *)$ generated by a^2, b^2, and $(ab)^2$, and prove your covering space is the correct one.

3. Describe a cell structure on the real projective space $\mathbb{RP}^n$. Explicitly describe the attaching maps of all cells. Use this cell structure to compute the homology groups of $\mathbb{RP}^n$ with coefficients in $\mathbb{Z}/2$.

4. Compute the homology groups of the cell complex obtained by identifying the sides of the polygon below as indicated:



5. Let M be closed, connected, 5-manifold such that $\pi_1(M) \cong \mathbb{Z}/7$ and $H_2(M) = \mathbb{Z}$. Compute all homology and cohomology groups of M with $\mathbb{Z}$-coefficients.

6. Let S^n be the n-sphere, for some integer $n \geqslant 1$, and $f: S^n \longrightarrow S^n$ a continuous map.

   (a) Show that if f is not surjective, then it is homotopic to a constant map.

   (b) Construct an example of a surjective map f that is homotopic to a constant map.

   (c) Is every map f as above homotopic to a constant? Either give a proof or a counterexample.