**MATH 6720**
**Qualifying exam**
**Due Date: 08/18/2022**

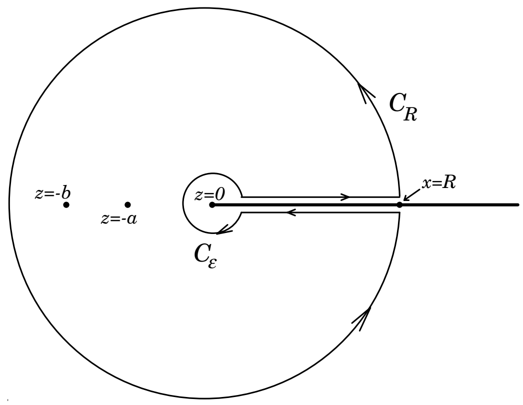
**Instructions**: There are five problems on the qualifying exam covering all the topics discussed in MATH 6720: complex numbers and elementary functions, analytic functions, series and singularities, complex integration, residue theory, conformal mappings and bilinear transformations, applications to fluid flow and PDEs, asymptotic series expansions of integrals.

You must solve **3 out 5** problems. Clearly indicate which problem should be graded, otherwise problems will be graded as they appear. Each problem is worth 25 points. A high pass corresponds to a score of 66 and above (out of 75). A pass corresponds to a score of between 57 and 65 (out of 75).

1. **Problem 1**:
   (a) (8 points) Let $P(z)$ be a polynomial of degree $n$, with $n$ simple roots, none of which lie on a simple closed contour $C$. Evaluate
   $$I = \frac{1}{2\pi i}\oint_C \frac{P'(z)}{P(z)}dz.$$
   (b) (8 points) Use Liouville's Theorem to prove the Fundamental Theorem of Algebra. Let $P(z)$ be any polynomial of integer degree $m \geq 1$. Then there is at least one point $z = \alpha$ such that $P(\alpha) = 0$; that is $P(z)$ has at least one root.
   (c) (9 points) Let $f(z)$ be a meromorphic function, i.e. a function that only has poles in the finite plane, defined inside and on a simple closed contour $C$, with no zeros or poles on $C$. Let $N$ and $P$ be the number of zeroes and poles, respectively, of $f(z)$ inside $C$; where a multiple zero or pole is counted according to its multiplicity. Show that
   $$I = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N - P.$$
2. **Problem 2**:
   Recall the binomial expansion
   $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k,$$
   where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
   (a) (12 points) Use the binomial expansion and Cauchy's Integral Theorem to evaluate
   $$\oint \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}$$
   where $C$ is the unit circle centered at the origin.
   (b) (13 points) Use this result to establish the following real integral formula:
   $$\frac{1}{2\pi}\int_0^{2\pi} (\cos\theta)^{2n} d\theta = \frac{(2n)!}{4^n (n!)^2}.$$
3. (25 points) **Problem 3**:
   Use residue calculus to show that
   $$\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^4+4}dx = \frac{\pi}{2}e^{-a}\sin(a),\ a > 0.$$

4. (25 points) **Problem 4**:
Use the keyhole contour below to show that on the principal branch of $x^k$

$$I(a) = \int_0^\infty \frac{x^{k-1}}{x+a} dx = \frac{\pi}{\sin(k\pi)} a^{k-1},\ 0 < k < 2,\ a > 0.$$




5. **Problem 5**:
Consider a source at $z = -a$ and a sink at $z = a$ of equal strength $k$.
   (a) (12 points) Show that the associated complex potential is $\Omega(z) = k \log[(z+a)/(z-a)]$.
   (b) (13 points) Show that the flow speed is $2ka/\sqrt{a^4 - 2a^2r^2\cos(2\theta) + r^4}$, where $z = re^{i\theta}$.