UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differentiable Manifolds

August, 2023.

Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points.

Problem 1. Let $\alpha : G \times M \to M$ be an action of a Lie group G on a C^∞ manifold M such that α is a continuous function. Prove or find a counterexample: if $\alpha\big|_{G\times\{x\}} : G \to M$ is a C^∞ immersion for every $x \in M$, then the function α is C^∞.

Problem 2. For which values of n is the manifold $\mathbb{RP}^n$ orientable? Prove that your answer is correct.

Problem 3. Let $R : S^2 \to \mathbb{R}_+$ be a nonconstant C^∞ function. Show that $F : S^2 \to \mathbb{R}^3$ is defined by $F(v) = R(v)v$, then there exists a value of $r \in \mathbb{R}_+$ such that if S^2_r is the sphere of radius r centered at 0, then the intersection of S^2_r with the image of F is a nonempty finite union of circles.

Problem 4. Let $V(x, y, z) = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ and $W(x, y, z) = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$ be vector fields on $\mathbb{R}^3$. Show that there does not exist a surface $S \subset \mathbb{R}^3$ such that both V and W are tangent to S.

Problem 5. Identify $\mathcal{M}(2)$, the set of two-by-two matrices, with $\mathbb{R}^4$. Let $SL(2,\mathbb{R}) \subset \mathcal{M}(2)$ be the set of matrices with determinant one. Show that $SL(2,\mathbb{R})$ is a smooth submanifold and calculate the tangent space $T_{id}SL(2,\mathbb{R})$ as a subspace of $T_{id}\mathcal{M}(2) = \mathbb{R}^4$.

Problem 6. Let ω be a closed n-form on a smooth manifold M. Let N be a compact, orientable n-manifold without boundary and $f\colon N \to M$ a smooth map with

$$\int_N f^*\omega \neq 0.$$

Show that deRham cohomology group $H^n_{dR}(M)$ is not trivial.