UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Algebraic Topology
August 18, 2023

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points.

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1. Let a and b be the two generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$, and prove this covering space is indeed the correct one.

2. Let X_n be the topological space obtained by attaching a disk D to the torus $T = S^1 \times S^1$ where the attaching map is a degree n map from ∂D to $S^1 \times \{p\}$ in T.
   (a) Calculate $\pi_1(X_n)$.
   (b) Calculate the homology and cohomology of X_n with $\mathbb{Z}$ coefficients.

3. Let Σ_g be the closed, orientable, genus-g surface. Show that $\pi_1(\Sigma_2)$ contains $\pi_1(\Sigma_4)$ as a normal subgroup.

4. Prove or disprove:
   (a) For every $d \in \mathbb{Z}$ there is a degree d map $f_d \colon T^2 \to S^2$.
   (b) For every $d \in \mathbb{Z}$ there is a degree d map $g_d \colon S^2 \to T^2$.

   Here S^2 is the 2-sphere and $T^2 = S^1 \times S^1$ is the 2-dimensional torus.

5. Let M be a closed, orientable 5-manifold with $H_1(M) = \mathbb{Z} \oplus \mathbb{Z}_5$ and $H_2(M) = \mathbb{Z}_2$. Find $H_i(M)$ and $H^i(M;\mathbb{Z})$ for all $i \geq 0$.

6. Construct a Δ-complex structure on $\mathbb{R}P^2$ and use it to calculate the cohomology groups $H^i(\mathbb{R}P^2;\mathbb{Z}_2)$ and the cup product structure.