

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH
Ph.D. Preliminary Examination: Applied Complex Variables and Asymptotic Methods
Fall 2023

This exam is closed book, closed notes, and no calculators are allowed. There is a formula sheet appended to this exam that you may use to complete the problems. You have two hours to complete this exam.

There are 5 problems below. You must complete 3 of them. Each problem is worth 10 points. Clearly indicate which 3 problems you wish to be graded; otherwise, the first 3 problems with work shown will be graded.

- A score of 24 (out of 30) is a *high pass*.
- A score of 20 (out of 30) is a *pass*.

1. Complete the following:

- (a) (5 pts) Prove that if both $f(z)$ and $\overline{f(z)}$ are analytic in some domain, then f is the constant function in that domain.
- (b) (5 pts) Identify, with justification, the branch points of the function $f(z) = \frac{1}{\sqrt{z+1}}$ and propose a corresponding branch cut.

2. Prove the argument principle: Assume that f is a meromorphic function inside and on a simple closed contour C (oriented in the typical counter-clockwise fashion), that f has exactly N zeros (counting multiplicity) in this region all lying within C , and that f has exactly M poles (counting order), all lying within C . Prove that

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N - M.$$

3. Let $f(z)$ be an entire function, and suppose there exists a non-negative constant C such that $|f(z)| \leq C|z|$ for every z . Prove that $f(z) = Az$, where A is a constant.

4. Evaluate the following integral by residue integration:

$$\int_0^\infty \frac{dx}{x^3 + a^3}, \quad a > 0.$$

Clearly indicate the contour of integration and the treatment of the integral along each segment.

5. Compute the following asymptotic expansions:

- (a) (5 pts) $I(k) = \int_k^\infty e^{-t^2} dt$ as $k \rightarrow 0^+$. You must compute the entire asymptotic expansion.
- (b) (5 pts) $I(k) = \int_0^2 e^{-k(t^2-3t)} dt$ as $k \rightarrow \infty$. You need only compute the leading order term in the expansion.

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Formula sheet

The Euler Gamma function for real inputs is defined as,

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0.$$

In what follows, C_R denotes a semicircular arc of radius R in the upper half-plane centered at the origin. The contour C_ϵ is a circular arc of radius ϵ centered around a point z_0 that sweeps out an angle of ϕ .

1. Suppose f is analytic on an open domain containing a simple closed loop C . Then for all integers $n \geq 0$ and all z enclosed by C ,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw,$$

2. The coefficients for a Laurent series of the function f are given by,

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$

3. If a continuous f is bounded over a contour C of finite length, i.e., $|f(z)| \leq M < \infty$ for all $z \in C$ and $\int_C |dz| = L < \infty$, then

$$\left| \int_C f(z) dz \right| \leq ML$$

4. Suppose $f(z) = P(z)/Q(z)$ is a rational function with $\deg Q \geq \deg P + 2$. Then,

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0.$$

5. (Jordan's Lemma) Suppose that $f(z) \rightarrow 0$ uniformly for $z \in C_R$ as $R \rightarrow \infty$. Then for any $k > 0$,

$$\lim_{R \rightarrow \infty} \int_{C_R} e^{ikz} f(z) dz = 0.$$

6. Suppose that $(z - z_0)f(z) \rightarrow 0$ uniformly for $z \in C_\epsilon$ as $\epsilon \rightarrow 0$. Then,

$$\lim_{\epsilon \rightarrow 0} \int_{C_\epsilon} f(z) dz = 0.$$

7. Suppose that f has a simple pole at $z = z_0$. Then

$$\lim_{\epsilon \rightarrow 0} \int_{C_\epsilon} f(z) dz = i\phi \text{Res}(f; z_0).$$

8. With C_R any origin-centered circular arc (not necessarily in the upper half-plane), if $zf(z) \rightarrow 0$ uniformly on C_R as $R \rightarrow \infty$, then,

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0.$$

Laplace-type integrals

These are formulas regarding asymptotic ($k \rightarrow \infty$) behavior of $I(k) := \int_a^b f(t)e^{-k\phi(t)} dt$ for $a < b$.

- (1) (Watson's Lemma) Set $a = 0$ and $\phi(t) = t$. Assume f is integrable with the series expansion,

$$f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n} \quad t \rightarrow 0^+, \quad \alpha > -1, \quad \beta > 0.$$

In addition, if $b < \infty$ then assume $|f(t)| \leq M < \infty$ for $t \in [a, b]$, and if $b = \infty$ then assume $f(t) = \mathcal{O}(e^{ct})$ as $t \rightarrow \infty$ for some $c \in \mathbb{R}$. Then,

$$I(k) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + \beta n + 1)}{k^{\alpha + \beta n + 1}}.$$

- (2) (Laplace's Method) Assume $b < \infty$, and that $\phi \in C^4([a, b])$ and $f \in C^2([a, b])$. Suppose that for some $c \in [a, b]$, we have $\phi'(c) = 0$ and $\phi''(c) > 0$. Also, assume that $\phi'(t) \neq 0$ for all $t \in [a, b] \setminus \{c\}$. Then,

$$I(k) \sim G(c)e^{-k\phi(c)} f(c) \sqrt{\frac{2\pi}{k\phi''(c)}} + \mathcal{O}\left(\frac{e^{-k\phi(c)}}{k^{G(c)+1/2}}\right), \quad G(c) := \begin{cases} 1, & c \in (a, b) \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

Fourier-type integrals

These are formulas regarding asymptotic ($k \rightarrow \infty$) behavior of $I(k) := \int_a^b f(t)e^{ik\phi(t)} dt$ for $a < b$.

- (1) Set $a = 0$, and $\phi(t) = \mu t$, where $\mu = \pm 1$, and $k > 0$. Suppose f vanishes infinitely smoothly at $t = b$, that $f \in C^\infty((0, b])$, and that for some $\gamma > -1$, $f(t) \sim t^\gamma + o(t^\gamma)$ as $t \rightarrow 0^+$. Then,

$$I(k) = \left(\frac{1}{k}\right)^{\gamma+1} \Gamma(\gamma + 1) e^{i\frac{\pi}{2}\mu(\gamma+1)} + o(k^{-(\gamma+1)}).$$

- (2) (Stationary phase) Suppose $c \in [a, b]$ is the only value of t where $\phi'(t)$ vanishes. Assume that f vanishes infinitely smoothly at both $t = a$ and $t = b$, and that both f and ϕ are C^∞ on the intervals $[a, c)$ and $(c, b]$. Suppose that there is some $\gamma > -1$ such that as $t \rightarrow c$,

$$\begin{aligned} \phi(t) - \phi(c) &\sim \alpha(t - c)^2 + o((t - c)^2), \\ f(t) &\sim \beta(t - c)^\gamma + o((t - c)^\gamma). \end{aligned}$$

Then with $\mu = \text{sgn } \alpha$,

$$\int_a^b f(t)e^{ik\phi(t)} dt \sim e^{ik\phi(c)} \beta \Gamma\left(\frac{\gamma + 1}{2}\right) e^{i\pi\frac{\gamma+1}{4}\mu} \left(\frac{1}{k|\alpha|}\right)^{\frac{\gamma+1}{2}} + o\left(k^{-\frac{\gamma+1}{2}}\right).$$