UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Real Analysis
August 2024.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points. Carefully state any theorems you use.

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1. Let
$$A = \left\{ \sum_{n=1}^{\infty} \frac{x_n}{3^n} \mid x_n = 0 \text{ or } 1 \right\} \subset [0,1]$$
Show that A has Lebesgue measure 0 but
$$A + A := \{x + y \mid x, y \in A\} = [0,1].$$

2. Suppose $f_i : \mathbb{R} \to \mathbb{R}$ is measurable for $i = 1, 2, \cdots$. Is the set
$$\{x \in \mathbb{R} \mid 2 \text{ is a limit point of } \{f_i(x)\}_{i=1}^{\infty}\}$$
measurable? A number t is a limit point of a sequence if some subsequence converges to it.

3. Prove the following generalization of the Dominated Convergence Theorem. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $f_n, g_n, f, g : X \to [0, \infty)$ are measurable and have finite integrals, $f_n \to f$, $g_n \to g$ (pointwise convergence), $f_n \le g_n$, $\int g_n \to \int g < \infty$. Then $\int f_n \to \int f$.

4. Let μ and ν be two finite positive measures on a measurable space $(X, \mathcal{M})$. Suppose for every $\epsilon > 0$, there exists $E \in \mathcal{M}$ such that $\mu(E) < \epsilon$ and $\nu(X \setminus E) < \epsilon$. Show that $\mu \perp \nu$ (μ and ν are mutually singular).

5. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Show that if $1 \le q < p \le \infty$ then $L^q(X) \supseteq L^p(X)$. Moreover, if X contains an infinite family of pairwise disjoint subsets of positive measure, then $L^q(X) \supsetneq L^p(X)$.

6. Prove that the dual space of $(c_0, \|\cdot\|_{\sup})$ is ℓ^1. Recall
$$c_0 = \{v \in \mathbb{R}^{\mathbb{N}} : \lim_{j\to\infty} v_j = 0\}$$
and $\ell^1 = \{w \in \mathbb{R}^{\mathbb{N}} : \sum_j |w_j| < \infty\}$ with $\|w\| = \sum_j |w_j|$.