UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Ordinary Differential Equations

January 14th, 2021.

Instructions: This examination consists of five problems, of which you should answer three. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth 20 points. A high-passing score is [50] and a passing score is [40].

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1. Let $f \in C^1(U, \mathbb{R}^n)$ for $U \subset \mathbb{R}^n$ and $x_0 \in U$. Given the Banach space $X = C([0,T], \mathbb{R}^n)$ with norm $\|x\| = \max_{0 \leq t \leq T} |x(t)|$, let

$$K(x)(t) = x_0 + \int_0^t f(x(s))ds$$

for $x \in X$. Define $V = \{x \in X \mid \|x - x_0\| \leq \epsilon\}$ for fixed $\epsilon > 0$ and suppose $K(x) \in V$ (which holds for sufficiently small T), so that $K : V \to V$ with V a closed subset of X.

(a) Using the fact that f is locally Lipschitz in U with Lipschitz constant L_0, and taking $x, y \in V$ show that

$$|K(x(t)) - K(y(t))| \leq L_0 t \|x - y\|.$$

Hence, show that

$$\|K(x) - K(y)\| \leq L_0 T \| \|x - y\| \quad x, y \in V.$$

(b) Choosing $T < 1/L_0$, apply the contraction mapping principle to show that the integral equation has a unique continuous solution $x(t)$ for all $t \in [0,T]$ and sufficiently small T. Hence establish existence and uniqueness of the initial value problem

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0.$$

(c) Show that the solution of the equation

$$\dot{x} = x^2, \quad x(0) = x_0$$

blows up in finite time. Does this contradict the result of part (b)?

2. Consider the T–periodic non-autonomous linear differential equation

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad A(t) = A(t+T)$$

Let $\Phi(t)$ be a fundamental matrix with $\Phi(0) = \mathbf{I}$.

(a) Show that there exists at least one nontrivial solution $\chi(t)$ such that

$$\chi(t+T) = \mu\chi(t)$$

where μ is an eigenvalue of $\Phi(T)$.

(b) Suppose that $\Phi(T)$ has n distinct eigenvalues μ_i, $i = 1, \ldots, n$. Show that there are then n linearly independent solutions of the form

$$x_i = p_i(t)\mathrm{e}^{\rho_i t}$$

where the $p_i(t)$ are T–periodic. How is ρ_i related to μ_i?

(c) Consider the equation $\dot{x} = f(t)A_0 x$, $x \in \mathbb{R}^2$, with $f(t)$ a scalar T–periodic function and A_0 a constant matrix with real distinct eigenvalues. Determine the corresponding Floquet multipliers.

(d) Suppose that the autonomous nonlinear equation $\dot{\mathbf{x}} = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, exhibits a limit cycle. By linearizing about this solution, explain how Floquet theory can be used to determine the linear stability of the limit cycle.

3. Consider the following linear equation for $\mathbf{x} \in \mathbb{R}^n$:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{t})\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

where $\mathbf{A}, \mathbf{B}(t)$ are $n \times n$ matrices. Suppose that all eigenvalues λ_j, $j = 1, \ldots, n$, of the matrix $\mathbf{A}$ satisfy $\mathrm{Re}(\lambda_j) < 0$, and let $\mathbf{B}(t)$ be continuous for $0 \leq t < \infty$ with $\int_0^\infty \|\mathbf{B}(t)\| dt < \infty$.

(a) Using the variation of constants formula show that there exist constants $K, \sigma > 0$ such that

$$|\mathbf{x}(t)| \leq K\mathrm{e}^{-\sigma(t-t_0)}|\mathbf{x}_0| + K\int_{t_0}^t \mathrm{e}^{-\sigma(t-s)}|\mathbf{x}(s)|\|\mathbf{B}(s)\|ds.$$

(b) Let $u(t) = \mathrm{e}^{\sigma t}|\mathbf{x}(t)|$, $v(t) = \|\mathbf{B}(t)\|$ and $c = K\mathrm{e}^{\sigma t_0}|\mathbf{x}_0|$. Show that the inequality of part (a) can be rewritten as

$$u(t) \leq c + \int_{t_0}^t v(s)u(s)ds.$$

(c) From Gronwall's inequality we have

$$u(t) \leq c\exp\left(\int_{t_0}^t v(s)ds\right).$$

Use this to show that the zero solution of the initial value problem is asymptotically stable.

4. (a) Consider the dynamical system

$$\begin{aligned}
\dot{x} &= -y + x(1 - z^2 - x^2 - y^2) \\
\dot{y} &= x + y(1 - z^2 - x^2 - y^2) \\
\dot{z} &= 0.
\end{aligned}$$

Determine the invariant sets and attracting set of the system. Give a general definition of the ω–limit set for a flow $\phi(x,t)$ in $\mathbb{R}^n$, and determine it in the case of a trajectory for which $|z(0)| < 1$.

(b) Use the Poincare-Bendixson Theorem and the fact that the planar system

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3$$

has only the one critical point at the origin to show that this system has a periodic orbit in the annular region $A = \{x \in R^2 \,|\, 1 < |x| < \sqrt{2}\}$.

(c) Give the definitions of Poincare and Liapunov stability. Show that solutions of the system $\dot{x} = y, \dot{y} = 0$ are Poincare but not Liapunov stable.

(d) Consider the system

$$\dot{x} = x - y - x(x^2 + y^2) + \frac{xy}{\sqrt{x^2 + y^2}}$$
$$\dot{y} = x + y - y(x^2 + y^2) - \frac{x^2}{\sqrt{x^2 + y^2}}.$$

Show that the above pair of equations can be rewritten in polar coordinates as

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 2\sin^2(\theta/2)$$

and sketch the phase portrait. Determine whether or not the fixed point $(1, 0)$ is Liapunov stable.

5. The simple pendulum consists of a point particle of mass m suspended from a fixed point by a massless rod of length L, which is allowed to swing in a vertical plane. If friction is ignored then the equation of motion is

$$\ddot{x} + \omega^2 \sin x = 0, \quad \omega^2 = \frac{g}{L},$$

where x is the angle of inclination of the rod with respect to the downward vertical and g is the gravitational constant.

(a) Using conservation of energy, show that the angular velocity of the pendulum satisfies

$$\dot{x} = \pm\sqrt{2}(C + \omega^2 \cos x)^{1/2},$$

where C is an arbitrary constant. Express C in terms of the total energy of the system.

(b) Sketch the phase diagram of the pendulum equation in the $(x, \dot{x})$-plane. Illustrate the one-parameter family of curves given by part (a) for different values of C. Take $-3\pi \leq x \leq 3\pi$. Indicate the fixed points of the system and the separatrices - curves linking the fixed points. Give a physical interpretation of the underlying trajectories in the two distinct dynamical regimes $|C| < \omega^2$ and $|C| > \omega^2$.

(c) Show that in the regime $|C| < \omega^2$, the period of oscillations is

$$T = 4\sqrt{\frac{L}{g}}K(\sin x_0/2),$$

where $\dot{x} = 0$ when $x = x_0$ and K is the complete elliptic integral of the first kind, which is defined by

$$K(\alpha) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - \alpha^2 \sin^2 u}} du.$$

HINT: Derive an integral expression for T and then perform a change of variables

$$\sin u = \frac{\sin(x/2)}{\sin(x_0/2)}.$$

(d) For small amplitude oscillations, the pendulum equation can be approximated by the linear equation

$$\ddot{x} + \omega^2 x = 0.$$

Solve this equation for the initial conditions $x(0) = A$, $\dot{x}(0) = 0$ and sketch the phase-plane for different values of A. Compare with the phase-plane for the full nonlinear equation in part (b).