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Instructions: There are five problems on the qualifying exam covering all the topics discussed in MATH 6720: complex numbers and elementary functions, analytic functions, series and singularities, complex integration, residue theory, conformal mappings and bilinear transformations, applications to fluid flow and PDEs, asymptotic series expansions of integrals. You can use the theorems listed below.

You must solve **4 out 5** problems. Clearly indicate which problem should be graded, otherwise problems will be graded as they appear. Each problem is worth 25 points. A high pass corresponds to a score of 88 and above. A pass corresponds to a score of between 76 and 87.

Integration Theorems:

1. Let $f(z) = N(z)/D(z)$ be a rational function such that the degree of $D(z)$ exceeds the degree of $N(z)$ by at least two. Then
$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0.$$

2. (Jordan's lemma) Suppose that on a circular arc C_R in the upper half plane we have $f(z) \to 0$ uniformly as $R \to \infty$. Then
$$\lim_{R\to\infty} \int_{C_R} e^{ikz} f(z)dz = 0 \quad k > 0.$$

3. Suppose that on the contour C_ε (small circular arc with radius ε centered at z_0 with arc with angle ϕ), we have $(z - z_0)f(z) \to 0$ uniformly as $\varepsilon \to 0$. Then
$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} f(z)dz = 0.$$

4. Suppose $f(z)$ has a simple pole at $z = z_0$ with residue $\text{Res}(f; z_0) = C_{-1}$. Then for the contour C_ε (small circular arc with radius ε centered at z_0 with arc with angle ϕ), we have
$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} f(z)dz = i\phi C_{-1}$$
where the integration is carried out in the positive sense.

5. If on a circular arc C_R of radius R and center $z = 0$, $zf(z) \to 0$ uniformly as $R \to \infty$, then
$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0.$$

Asymptotic Expansion Theorems:

1. (Watson's lemma) Consider the integral $I(k) = \int_0^b f(t)e^{-kt}dt$ for $b > 0$. Suppose that $f(t)$ is integrable on $(0, b)$ and that it has the asymptotic series expansion
$$f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n} \qquad t \to 0^+ \qquad \alpha > -1 \qquad \beta > 0.$$

Then

$$I(k) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha+\beta n+1)}{k^{\alpha+\beta n+1}} \qquad k \to \infty.$$

If b is finite, we must have that for $t > 0$, $|f(t)| \le A$ and if $b = \infty$, we must have $|f(t)| \le Me^{ct}$.

2. (Laplace's method) Consider the integral $I(k) = \int_a^b f(t)e^{-k\phi(t)}dt$. Assume that $\phi'(c) = 0, \phi''(c) > 0$ for $c \in [a,b]$. Assume that $\phi'(t) \neq 0$ in $[a,b]$ except at $t = c$, $\phi \in C^4[a,b]$ and $f \in C^2[a,b]$. If c is an interior point, then

$$I(k) \sim e^{-k\phi(c)} f(c) \sqrt{\frac{2\pi}{k\phi''(c)}} + O\left(\frac{e^{-k\phi(c)}}{k^{3/2}}\right) \qquad k \to \infty$$

If c is an endpoint, then the leading term is half that obtained when c is an interior point, and the error is $O(e^{-k\phi(c)}/k)$.

3. Consider $I(k) = \int_0^b f(t)e^{ik\mu t}dt$ with $b > 0$, $\mu = \pm 1$ and $k > 0$. Suppose that $f(t)$ vanishes infinitely smoothly at $t = b$ and that $f(t)$ and all its derivatives exist in $(0, b]$. Assume that $f(t) \sim t^\gamma + o(t^\gamma)$ as $t \to 0^+$, $\gamma \in \mathbb{R}$ and $\gamma > -1$. Then

$$I(k) = \left(\frac{1}{k}\right)^{\gamma+1} \Gamma(\gamma+1) e^{i\frac{\pi}{2}\mu(\gamma+1)} + o(k^{-(\gamma+1)}) \qquad k \to \infty.$$

4. Consider $I(k) = \int_a^b f(t)e^{ik\phi(t)}dt$. Assume that $c \in [a,b]$ is the only point in $[a,b]$ where $\phi'(t)$ vanishes. Assume that $f(t)$ vanishes infinitely smoothly at $t = a$ and $t = b$ and that both f, ϕ are infinitely differentiable on $[a,c)$ and $(c,b]$. Assume that $\phi(t)-\phi(c) \sim \alpha(t-c)^2+o((t-c)^2)$, $f(t) \sim \beta(t-c)^\gamma + o((t-c)^\gamma)$ as $t \to c$ and $\gamma > -1$. Then with $\mu = \text{sgn } \alpha$

$$\int_a^b f(t)e^{ik\phi(t)}dt \sim e^{ik\phi(c)} \beta \Gamma\left(\frac{\gamma+1}{2}\right) e^{i\pi\frac{\gamma+1}{4}\mu} \left(\frac{1}{k|\alpha|}\right)^{\frac{\gamma+1}{2}} + o\left(k^{-\frac{\gamma+1}{2}}\right) \qquad k \to \infty.$$

5. Consider $I(k) = \int_C f(z)e^{k\phi(z)}dz$. Consider a single path of steepest descent from a saddle point z_0 of order $n-1$. Assume that $\phi(t) - \phi(z_0) \sim \frac{(z-z_0)^n}{n!}\phi^n(z_0)$ as $z \to z_0$ and $\phi^n(z_0) = |\phi^n(z_0)|e^{i\alpha}$. Assume that $f(z) \sim f_0(z-z_0)^{\beta-1}$ as $z \to z_0$ and Re $\beta > 0$. Then

$$I(k) \sim \frac{f_0(n!)^{\frac{\beta}{n}} e^{i\beta\theta}}{n} \frac{e^{k\phi(z_0)}\Gamma\left(\frac{\beta}{n}\right)}{(k|\phi^n(z_0)|)^{\frac{\beta}{n}}}.$$

1. **Problem 1**: Analytic functions
   (a) (12 points) Let $u(x,y) = \frac{y}{x^2+y^2}$ be the real part of an analytic function $f(z)$. Find the imaginary part $v(x,y)$ and the function $f(z)$.
   (b) (13 points) Let $f(z), g(z)$ be two entire functions with no zeroes such that $\lim_{z\to\infty} \dfrac{f(z)}{g(z)} = 1$. Use Liouville's Theorem to show that $f(z) \equiv g(z)$ for all z.

2. **Problem 2**: Bilinear transformations
   Consider the transformation on the extended complex plane $\mathbb{C}_\infty$
   $$f(z) = w = z + \frac{1}{z}.$$
   (a) (7 points) Find all possible fixed points.
   (b) (9 points) Show that the image of the points in the upper half z-plane that are exterior to the circle $|z| = 1$ corresponds to the entire upper half w-plane.
   (c) (9 points) Find the image of the unit circle.

3. **Problem 3**: Series and singularities
   (a) (13 points) Expand the function $f(z) = \frac{z}{(z-2)(z+i)}$ in a Laurent series in powers of z in $1 < |z| < 2$.
   (b) (12 points) Discuss the type of singularity (removable, pole and order, essential, branch, cluster). If the type is a pole give the strength of the pole, and give the nature (isolated or not). Include the point at infinity.
   $$f(z) = \frac{e^{2z} - 1}{z^2}$$

4. (25 points) **Problem 4**: Residue theory
   Show that
   $$\int_0^\infty \frac{\cos(2x)}{(1+x^2)^2} dx = \frac{3\pi e^{-2}}{4}.$$

5. (25 points) **Problem 5**: Asymptotic expansions
   Find the complete asymptotic expansion of
   $$I(k) = \int_0^\infty e^{-kt} t^{-1/3} \cos t dt \qquad k \to \infty.$$