Math 6520 Qualifying Exam, May 2022

Answer at most five of the problems below. If you answer more than five problems, let me know which of the five you would like me to grade. Complete solutions to three problems will be a high pass. Show all your work, and provide reasonable justification for your answers. ***Please state carefully any (major) results you use.***

1. Let W be the union of the three coordinate axes in $\mathbb{R}^3$ and set $X := \mathbb{R}^2 \setminus W$; thus X consists of points (x,y,z) where none of the coordinates are zero. Compute $\pi_1(X)$.

2. Let $f\colon M \longrightarrow N$ be a covering map between manifolds that are both nonempty, connected, and compact without boundary. Suppose f is homotopic to a constant map. Prove that N is a point.

3. Let X be a CW complex consisting of one vertex p, two edges a and b attached to p along their boundaries, and two faces f_1 and f_2, attached along ab^2 and ba^2, respectively.

   (a) Find $\pi_1(X)$. Is it a finite group?

   (b) Compute the homology groups $\mathrm{H}_i(X;\mathbb{Z})$ of X for all i.

4. Let T^2 be the real 2-dimensional torus and $f\colon S^1 \vee S^1 \longrightarrow T^2$ a continuous map. Prove or disprove the following assertions:

   (a) There is a continuous map $g\colon T^2 \longrightarrow S^1 \vee S^1$ such that $f \circ g\colon T^2 \longrightarrow T^2$ is homotopy to the identity map.

   (b) There is a continuous map $g\colon T^2 \longrightarrow S^1 \vee S^1$ such that $g \circ f$ is homotopy to the identity map.

   Justify your answers.

5. Prove that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$.

6. Let M be an orientable n-manifold that is connected, compact, and without boundary. Let $f\colon S^n \longrightarrow M$ be a continuous map such that the induced map $f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}) \longrightarrow \mathrm{H}_n(M;\mathbb{Z})$ is non-trivial. Calculate $\mathrm{H}^i(M;\mathbb{Q})$ and $\mathrm{H}_i(M;\mathbb{Q})$ for all i.