Department of Mathematics, University of Utah
Ph.D. Preliminary Examination: Analysis of Numerical Methods, II
Spring 2024

This exam is closed book, closed notes, and no calculators are allowed. You have two hours to complete this exam.

There are 5 problems below. You must complete 3 of them. Each problem is worth 20 points. Clearly indicate which 3 problems you wish to be graded; otherwise, the first 3 problems with work shown will be graded.

- A score of 52 (out of 60) is a *high pass*.
- A score of 48 (out of 60) is a *pass*.

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1. (20 pts)

(a) (10 pts) Compute coefficients A, B, C in the 3-point one-sided approximation

$$u''(x) \approx Au(x) + Bu(x+h) + Cu(x+2h)$$

that has optimal h-order of accuracy. What is this order of accuracy?

(b) (10 pts) Compute weights for the following quadrature rule that makes it exact for all polynomials of degree 2.

$$\int_{-1}^{1} f(x)\mathrm{d}x = w_0 f(0) + w_1 f'(0) + w_2 f''(0),$$

2. (20 pts)

(a) (10 pts) Compute the stability/amplification factor for forward Euler to solve $u' = f(t,u)$, and plot the region of the stability.

(b) (10 pts) Compute the stability/amplification factor for the trapezoidal rule/Crank-Nicolson scheme and use this to show that the scheme is A-stable.

3. (20 pts) Consider the multi-step method,

$$u_{n+1} + \alpha_1 u_n = k\beta_0 f_{n+1} + k\beta_1 f_n,$$

where $u_n \approx u(t_n)$, $f_n = f(t_n, u_n)$, $k = t_{n+1} - t_n$, and $u' = f(t,u)$.

(a) (10 pts) Identify the coefficients α, β that yield a scheme of optimal k-order of accuracy, and identify this order of accuracy.

(b) (10 pts) Determine whether or not this scheme is 0-stable and/or A-stable.

4. (20 pts)

(a) (10 pts) Use von Neumann stability analysis to determine a stability condition for,

$$D^+ u_j^n = D_0 u_j^n,$$

where $D^+ u_j^n = (u_j^{n+1} - u_j^n)/k$, $D_0 u_j^n = (u_{j+1}^n - u_{j-1}^n)/(2h)$, $D_\pm u_j^n = \pm(u_{j\pm1}^n - u_j^n)/h$.

(b) (10 pts) Use von Neumann stability analysis to determine a stability condition for

$$D^+ u_j^n = D_+ D_- u_j^n.$$

5. (20 pts) For the ODE $\boldsymbol{u}' = \boldsymbol{f}(t, \boldsymbol{u})$ with initial condition $\boldsymbol{u}(0)$ and $\boldsymbol{f}$ globally Lipschitz continuous in $\boldsymbol{u} \in \mathbb{R}^M$ uniformly in t, show that forward Euler with initial state $\boldsymbol{u}_0 = \boldsymbol{u}(0)$ is *convergent* to first order. A possibly helpful definition: a scheme is 0-stable if,

$$\max_{n \in [N]} \|\boldsymbol{e}_n\| \le C \max_{n \in [N]} \|R_n(\boldsymbol{u}(t_n))\|,$$

where $\boldsymbol{e}_n$ is the time-t_n error, and $R_n(\boldsymbol{u}(t_n))$ is the scheme residual using the exact solution.