8.5 Applications of Matrices and Determinants

You will learn to

- Use Cramer's rule to solve a system by determinants.
- Determine the area of a triangle given three vertices on the coordinate plane.
- Write an equation of a line given two points.

Cramer's Rule

1st col 2nd col

① If $ax + by = c$ then $x =$ $y=$

② $dx + ey = f$

① $by = c - ax$

$y = \frac{c-ax}{b}$

② $dx + e\left(\frac{c-ax}{b}\right) = f$

$bdx + ec - aex = bf$

$x(bd-ae) = bf - ce \Rightarrow$

① $y = \frac{c - a\left(\frac{bf-ce}{bd-ae}\right)}{b}$

$y = \frac{c\cancel{b}d - \cancel{aec} - a\cancel{b}f + \cancel{ace}}{\cancel{b}(bd-ae)}$

$$\boxed{x = \frac{bf-ce}{bd-ae} \qquad y = \frac{cd-af}{bd-ae}}$$

We can rewrite this as Cramer's rule:

$$\boxed{x = \frac{D_x}{D}}$$

$$y = \frac{D_y}{D}$$

$$x = \frac{-\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{-\begin{vmatrix} a & b \\ d & e \end{vmatrix}} = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

Notice

$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd$

$\begin{vmatrix} c & b \\ f & e \end{vmatrix} = ce - bf$

$\begin{vmatrix} a & c \\ d & f \end{vmatrix} = af - cd$

Example 1 Use Cramer's rule to solve this :

$5x - 2y = 3$
$6x + 4y = -8$

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 3 & -2 \\ -8 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ 6 & 4 \end{vmatrix}} = \frac{12-16}{20-{}^{-}12} = \frac{-4}{32} = \frac{-1}{8}$$

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 5 & 3 \\ 6 & -8 \end{vmatrix}}{32} = \frac{{}^{-}40-18}{32} = \frac{{}^{-}58}{32} = \frac{-29}{16}$$

Soln: $\left(\frac{-1}{8}, \frac{-29}{16}\right)$

Cramer's rule can be used to solve a 3 x 3 system as well.

Example 2:

Set up the determinants for this system:

$$\begin{aligned} -y + 2z &= 3 \\ 4x + y &= 5 \\ x - 2z &= -6 \end{aligned}$$

$D = \begin{vmatrix} 0 & -1 & 2 \\ 4 & 1 & 0 \\ 1 & 0 & -2 \end{vmatrix}$ -10

$D_x = \begin{vmatrix} 3 & -1 & 2 \\ 5 & 1 & 0 \\ -6 & 0 & -2 \end{vmatrix}$ -4

$D_y = \begin{vmatrix} 0 & 3 & 2 \\ 4 & 5 & 0 \\ 1 & -6 & -2 \end{vmatrix}$ -34

$D_z = \begin{vmatrix} 0 & -1 & 3 \\ 4 & 1 & 5 \\ 1 & 0 & -6 \end{vmatrix}$ -32

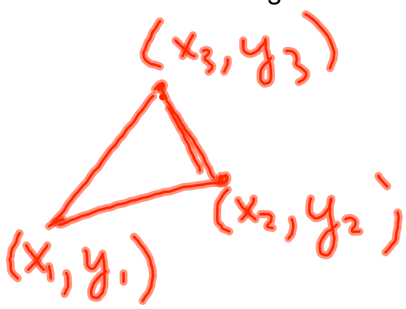
$x = \frac{-4}{-10} = \frac{2}{5}$

$y = \frac{-34}{-10} = \frac{17}{5}$

$z = \frac{-32}{-10} = \frac{16}{5}$

$\left(\frac{2}{5}, \frac{17}{5}, \frac{16}{5}\right)$

Area of a triangle with vertices (x_1, y_1) , (x_2, y_2) , (x_3, y_3)



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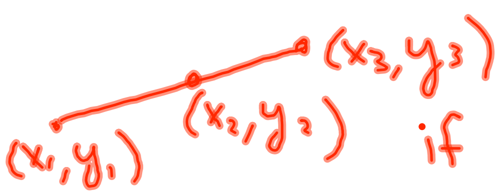
$$A = \pm \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Test for collinearity (all pts on same line)

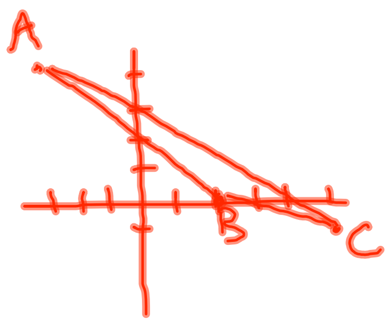
triangle w/ zero area



if $0 = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ then pts are collinear

Example 2: Determine if these three points are collinear. If not, then find the area of the triangle which has them as the three vertices.

A (-3,4) B (2,0) C (5, -1)



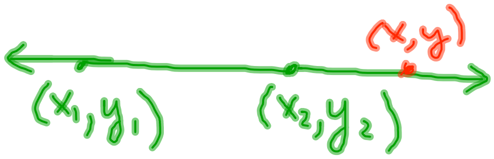
$$\begin{vmatrix} -3 & 4 & 1 \\ 2 & 0 & 1 \\ 5 & -1 & 1 \end{vmatrix} \begin{matrix} -3 & 4 \\ 2 & 0 \\ 5 & -1 \end{matrix} = 0 + 20 + -2 - 8 - 3 - 0 = 7$$

$\Rightarrow$ A, B, C are <u>not</u> collinear

$$\text{Area} = \pm \frac{1}{2}(7) = \frac{7}{2}$$

Two point form of the equation of a line.

An equation of a line through the points (x_1, y_1) and (x_2, y_2) can be found using determinants.



$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$x_1\ y_1 \quad x_2\ y_2$

Example 3: Write an equation of the line through the points (1, 5) and (0, -2)

$$\begin{array}{c} + \\ - \\ \end{array}\begin{vmatrix} 1 & 5 & 1 \\ 0 & -2 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$-0 + -2\begin{vmatrix} 1 & 1 \\ x & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 5 \\ x & y \end{vmatrix} = 0$$

$$-2(1-x) - (y-5x) = 0$$

$$-2 + 2x - y + 5x = 0$$

$$-2 + 7x = y$$

$$\boxed{y = 7x - 2}$$