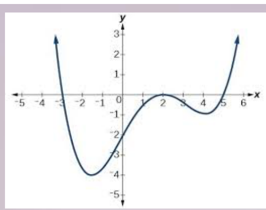

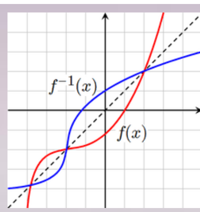

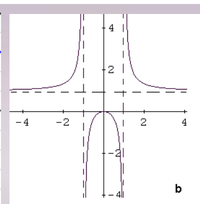

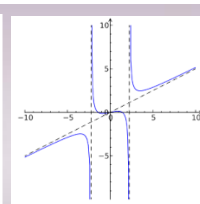

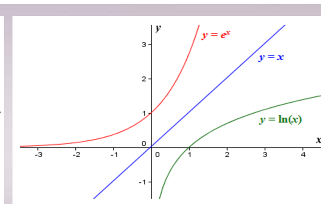

Math 1050 ~ College Algebra

26 Systems of Linear Equations: Determinants

$$-3x + 4y = 5$$
$$2x - y = -10$$
$$\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$
$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

Learning Objectives

- Find the determinant of a 2×2 or 3×3 matrix.
- Solve a system of linear equations using Cramer's Rule.

Determinant of a Matrix

Every square matrix has a number associated with it, called the determinant of A. It may be written det(A) or $|A|$.

For a 2 × 2 matrix, det(A) is given by this formula.

$$\boxed{Det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.}$$

NOTE det(A) is a scalar, not a matrix.

Ex 1: Find the determinant of each of these matrices.

a) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

$\begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = 2(3)-5(1) = 1$

b) $\begin{bmatrix} 3 & 5 \\ 2 & 8 \end{bmatrix}$

$\begin{vmatrix} 3 & 5 \\ 2 & 8 \end{vmatrix} = 3(8)-2(5) = 14$

c) $\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$ this matrix has no inverse

$\begin{vmatrix} 3 & -1 \\ -6 & 2 \end{vmatrix} = 6-(6) = 0$

Cramer's Rule

For a set of two equations in two unknowns, Cramer's Rule says that

$$\boxed{\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned} \text{ has solutions } x = \frac{ce - bf}{ae - bd},\ y = \frac{af - cd}{ae - bd}}.$$

to have solution, $ae - bd \neq 0$.

Notice: $ae-bd = \begin{vmatrix} a & b \\ d & e \end{vmatrix}$ determinant of coefficient matrix for the system of eqns.

Ex 2: Use the rule above to determine the solution.

$$\begin{aligned} 2x + y &= 4 \\ 5x + 3y &= -1 \end{aligned}$$

$x = \dfrac{\begin{vmatrix} 4 & 1 \\ -1 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix}}$ $\qquad$ $y = \dfrac{\begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix}}$

$x = \dfrac{12-(-1)}{6-5}$ $\qquad$ $y = \dfrac{-2-20}{6-5}$

$x = 13$ $\qquad$ $y = -22$

pt of intersectn (solution) is $\boxed{(13,-22)}$

Determinant of a 3×3 matrix is more complex.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \det(A) = |A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

subscripts on a tell the location of that element in the matrix, with row first and column second

Given the square $n \times n$ matrix A where $n > 1$, and a_{ij} represents the entry in the i^{th} row and j^{th} column:

- the minor, M_{ij} of the entry a_{ij} is the determinant of the $(n\text{-}1)\times(n\text{-}1)$ matrix left after deleting row i and column j from the matrix A.
- the cofactor, C_{ij} of entry a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Ex 3: Find all M_{ij} and C_{ij} for this matrix. $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

$M_{11} = \begin{vmatrix} 0 & -1 \\ -2 & -3 \end{vmatrix} = 0-(2) = -2$

$M_{12} = \begin{vmatrix} 1 & -1 \\ 6 & -3 \end{vmatrix} = -3-(-6) = 3$

$M_{13} = \begin{vmatrix} 1 & 0 \\ 6 & -2 \end{vmatrix} = -2-0 = -2$

$M_{21} = \begin{vmatrix} -1 & 0 \\ -2 & -3 \end{vmatrix} = 3-0 = 3$

$M_{22} = \begin{vmatrix} 1 & 0 \\ 6 & -3 \end{vmatrix} = -3-0 = -3$

$M_{23} = \begin{vmatrix} 1 & -1 \\ 6 & -2 \end{vmatrix} = -2-(-6) = 4$

$M_{31} = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1$ $\quad M_{32} = \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = -1$ $\quad M_{33} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1$

$C_{11} = (-1)^2(-2) = -2$, $C_{12} = (-1)^3(3) = -3$, $C_{13} = (-1)^4(-2) = -2$

$C_{21} = (-1)^3(3) = -3$, $C_{22} = (-1)^4(-3) = -3$, $C_{23} = (-1)^5(4) = -4$

$C_{31} = (-1)^4(1) = 1$, $C_{32} = (-1)^5(-1) = 1$, $C_{33} = (-1)^6(1) = 1$

The determinant of an $n \times n$ matrix, where n > 1, is the sum of the entries in any row or column multiplied by each entry's respective cofactor.

Ex 4: Find the determinant of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$.

way 1: (choose row 1)

$$|A| = 1\begin{vmatrix} 0 & -1 \\ -2 & -3 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -1 \\ 6 & -3 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 6 & -2 \end{vmatrix}$$

$$= 1(0-2) + 1(-3-(-6))$$

$$= 1(-2) + 3 = 1$$

way 2: (choose column 2)

$$|A| = (-1)(-1)\begin{vmatrix} 1 & -1 \\ 6 & -3 \end{vmatrix} + (1)\,0\begin{vmatrix} 1 & 0 \\ 6 & -3 \end{vmatrix} + (-1)(-2)\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}$$

$$= 1(-3-(-6)) + 0(-3-0) + 2(-1-0)$$

$$= 3 + 0 + -2 = 1$$

To use Cramer's Rule to solve a set of 3 equations, let $D = \det A$. D_x is found by replacing the first column of A by the constants. D_y is found by replacing the second column of A by the constants, and D_z is found by replacing the third column of A by the constants.

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}$$

Ex 5: Use Cramer's Rule to solve.

$$\begin{aligned} x - y &= 1 \\ x - z &= -2 \\ 6x - 2y - 3z &= -4 \end{aligned}$$

const. matrix $\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$ note: this is same matrix as last example.

row 1

$$x = \frac{\begin{vmatrix} 1 & -1 & 0 \\ -2 & 0 & -1 \\ -4 & -2 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{vmatrix}} = \frac{1\begin{vmatrix} 0 & -1 \\ -2 & -3 \end{vmatrix} - (-1)\begin{vmatrix} -2 & -1 \\ -4 & -3 \end{vmatrix} + 0}{1} = 1(-2) + 1(2) = \boxed{0}$$

column 3

$$y = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 1 & -2 & -1 \\ 6 & -4 & -3 \end{vmatrix}}{1} = +0 - (-1)\begin{vmatrix} 1 & 1 \\ 6 & -4 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= 1(-4-6) - 3(-2-1) = -10 + 9 = -1$$

row 2

$$z = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 6 & -2 & -4 \end{vmatrix}}{1} = -1\begin{vmatrix} -1 & 1 \\ -2 & -4 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 6 & -4 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -1 \\ 6 & -2 \end{vmatrix}$$

$$= -1(4+2) + 0 + 2(-2+6) = -6 + 8 = 2$$

$\Rightarrow$ solution (pt of intersection) is $\boxed{(0, -1, 2)}$