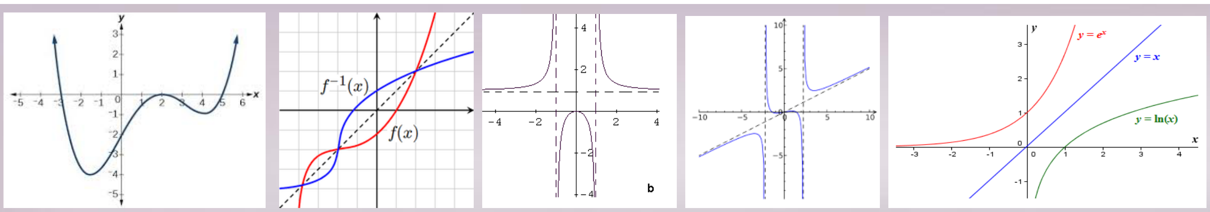


Math 1050 ~ College Algebra

28 Sequences

$$-3x + 4y = 5$$
$$2x - y = -10$$
$$\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$
$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

Learning Objectives

- Identify number patterns.
- Recognize and use recursive and explicit formulas for sequences.
- Graph sequences.
- Identify arithmetic and geometric sequences.
- Find formulas for arithmetic and geometric sequences.

Number Patterns

An ordered collection of numbers or events is called a sequence. There are many interesting numeric sequences. If it goes on indefinitely, it is called an infinite sequence.

Ex 1: For each sequence, determine how to find the next term and state two more terms.

a) 2, 4, 8, ... next terms: 16, 32 (this sequence is powers of 2)

$\times 2$ $\times 2$

$+1$ $+1$ $+1$

b) $\frac{2}{9}, \frac{3}{8}, \frac{4}{7}, \frac{5}{6}, \ldots$ next terms: $\frac{6}{5}, \frac{7}{4}$

-1 -1 -1

c) 2, 4, 16, ... next terms: $16^2, (16^2)^2$

$4=2^2$, $16=4^2$ or 256, 65536

d) 19, 11, 3, -5, -13, ... next terms: -21, -29

-8 -8 -8 -8

e) 16, -8, 4, -2, ... next terms: 1, $-\frac{1}{2}$

$\times -\frac{1}{2}$ $\times -\frac{1}{2}$ $\times -\frac{1}{2}$

$+$ $+$ $+$ $+$ $+$

f) 1, 1, 2, 3, 5, 8, ... next terms: 13, 21

(Fibonacci sequence)

g) 27, 18, 12, 8, ... next terms: $\frac{16}{3}, \frac{32}{9}$

$\times \frac{2}{3}$ $\times \frac{2}{3}$ $\times \frac{2}{3}$

A recursive formula defines each new term by one or more of the previous terms.

$a_1\ a_2\ a_3\ a_4\ a_5\ a_6$

1, 1, 2, 3, 5, 8, ...

n is a counter (input variable)

a_n is the n^{th} term of the sequence

$$\boxed{a_n = a_{n-1} + a_{n-2},\ a_1=1,\ a_2=1}$$

to get to next term,

we add 2 previous terms

ex: $a_5 = a_4 + a_3$

An explicit formula describes how to find any term of the sequence.

$a_1\ a_2\ a_3$

2, 4, 8, ...

$\boxed{a_n = 2^n}$, $n=1,2,3,\ldots$

each term is 2 to a power

ex what is a_{100}? $a_{100} = 2^{100}$

Ex 2: Write five terms for each sequence. Identify as recursive or explicit.

explicit (assume n starts at 1)

a) $a_n = (-1)^n\left(\frac{n}{n+1}\right)^n$

$a_1 = -\left(\frac{1}{2}\right)^1 = \frac{-1}{2}$

$a_2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

$a_3 = -\left(\frac{3}{4}\right)^3 = \frac{-27}{64}$

$a_4 = \left(\frac{4}{5}\right)^4$

$a_5 = -\left(\frac{5}{6}\right)^5$

recursive

b) $a_n = \frac{1}{a_{n-1}}$, $a_1 = 1$

$a_1 = 1$

$a_2 = \frac{1}{a_1} = \frac{1}{1} = 1$

$a_3 = \frac{1}{a_2} = 1$

$a_4 = a_5 = 1$

what if $a_1 = 2$?

$a_2 = \frac{1}{2}$

$a_3 = \frac{1}{1/2} = 2$

$a_4 = \frac{1}{2}$

$a_5 = 2$

Ex 3: Write an explicit and a recursive formula for this sequence.

2, -4, 6, -8, ...

let's work with sequence 2,4,6,8,... first.

recursive: $a_n = a_{n-1} + 2,\ a_1 = 2$

explicit: $a_n = 2n \quad n=1,2,3,\ldots$

2, -4, 6, -8, ...

recursive: $a_n = (-1)^{n+1}\left(|a_{n-1}| + 2\right),\ a_1 = 2$

explicit: $a_n = (-1)^{n+1}(2n)$

n	a_n
1	2
2	-4
3	6
4	-8

A sequence is a function with the domain of all natural numbers or a consecutive subset of the natural numbers.

Here are different ways to describe odd natural numbers as a sequence. Note that the fourth option allows us to have a finite sequence or one beginning with 0.

① $f(n) = 2n - 1,\ n = 1, 2, 3, \ldots$ $f(1) = 2(1) - 1 = 1,\ f(2) = 3,\ f(3) = 5, \ldots$

② $a_n = 2n - 1,\ n = 1, 2, 3, \ldots$

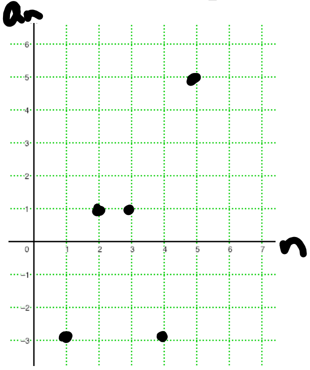
③ $\{2n - 1\}_{n=1}^{\infty}$

④ $\{2n - 1\}$

Since a sequence is a function, we can graph it. Note that the graph will be points, not connected with a curve.

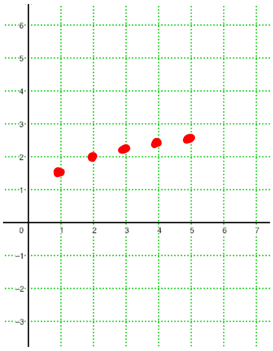
Ex 4: Write five terms of each sequence and plot each on the graph.

a) $\{(-1)^n(5 - 2n)\}_{n=1}^{\infty}$



n	a_n
1	-3
2	1
3	$-(-1) = 1$
4	-3
5	$-(-5) = 5$

b) $\left\{\frac{3n}{n+1}\right\}_{n=1}^{\infty}$



n	a_n
1	$\frac{3}{2} = 1.5$
2	$\frac{6}{3} = 2$
3	$\frac{9}{4} = 2.25$
4	$\frac{12}{5} = 2.4$
5	$\frac{15}{6} = \frac{5}{2} = 2.5$

Arithmetic Sequence

$\{a_n\}$ is an arithmetic sequence if successive terms have the same difference.

recursive formula

$a_n = a_{n-1} + d$ with a_1 given

d constant

explicit/direct formula $\boxed{a_n = a_1 + (n-1)d}$

Ex 4: Which of these sequences are arithmetic? For those that are, find d and a_{20}. (assume n starts at 1)

a) 5.3, 5.7, 6.1, 6.5, 6.9, ...

+0.4 +0.4 +0.4 +0.4

$d = 0.4$

$a_n = 5.3 + (n-1)(0.4)$

$a_{20} = 5.3 + 19(0.4) = 12.9$

b) ln 2, ln 5, ln 8, ln 11, ...

(aside: 2, 5, 8, 11, . . . arith. seq.)

+3 +3 +3

given seq. is NOT arithmetic

c) 800, 400, 200, 100, 50, ...

$\times\frac{1}{2}$ $\times\frac{1}{2}$ $\times\frac{1}{2}$ $\times\frac{1}{2}$

NOT arithmetic

d) $\frac{19}{2}, \frac{11}{2}, \frac{3}{2}, -\frac{5}{2}, -\frac{13}{2}, \ldots$

(aside: just numerators

19, 11, 3, -5, -13, . . . is arith. seq.

-8 -8 -8

d = -8)

original seq. is also arithmetic

$d = -4$, $a_n = \frac{19}{2} + (n-1)(-4) \Rightarrow a_{20} = \frac{19}{2} + 19(-4)$

$a_{20} = -66.5$

Ex 5: Find an explicit formula for a_n, such that $\{a_n\}$ is arithmetic and $a_1 = 0$, $d = -2/3$. Write the first five terms.

$a_n = a_1 + (n-1)d$

$a_n = 0 + (n-1)(\frac{-2}{3})$

$a_n = \frac{2}{3} - \frac{2}{3}n$

n	a_n
1	0
2	$\frac{-2}{3}$
3	$-4/3$
4	$-6/3 = -2$
5	$-8/3$

$0, \frac{-2}{3}, \frac{-4}{3}, -2, \frac{-8}{3}, \ldots$

Ex 6: Find an explicit formula for the arithmetic sequence such that $a_2 = 3$ and $a_7 = 33$.

way 1 $a_n = a_1 + (n-1)d$

① $3 = a_1 + (2-1)d$

$3 = a_1 + d$

② $33 = a_1 + (7-1)d$

$33 = a_1 + 6d$

use substitution:

① $a_1 = 3 - d$

$\Rightarrow$ ② $33 = (3-d) + 6d$

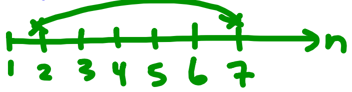
$30 = 5d$

$d = 6 \Rightarrow a_1 = 3 - 6 = -3$

$\boxed{a_n = -3 + (n-1)6}$

way 2

33-3=30



$\frac{30}{5 \text{ jumps from 2 to 7}} = 6 = d$

$d = 6 \Rightarrow a_1 = a_2 - 6$

$a_1 = 3 - 6 = -3$

Geometric Sequence

$\{a_n\}$ is a geometric sequence if successive terms have the same quotient (ratio).

$a_n = a_{n-1}r$ with a_1 given

recursive formula

explicit formula: $\boxed{a_n = a_1\left(r^{n-1}\right)}$

Ex 7: Which of these are geometric? If they are, determine r and find a_{10}.

a) $\frac{1}{5}, \frac{2}{7}, \frac{3}{9}, \frac{4}{11}, \ldots$

NOT geom.

(also not arithmetic)

b) $\frac{19}{2}, \frac{11}{2}, \frac{3}{2}, -\frac{5}{2}, -\frac{13}{2}, \ldots$

NOT geometric

(it's arithmetic instead)

c) 800, 400, 200, 100, 50, ...

$\times\frac{1}{2}$ $\times\frac{1}{2}$ $\times\frac{1}{2}$ $\times\frac{1}{2}$

IS geometric, $r = \frac{1}{2}$

$a_n = 800\left(\frac{1}{2}\right)^{n-1} \Rightarrow a_{10} = 800\left(\frac{1}{2}\right)^{9} = \frac{25}{16}$

d) $9, -6, 4, -\frac{8}{3}, \ldots$

$\times -\frac{2}{3}$ $\times -\frac{2}{3}$ $\times -\frac{2}{3}$

IS geometric

$r = -\frac{2}{3}$

$a_n = 9\left(-\frac{2}{3}\right)^{n-1}$

$a_{10} = 9\left(-\frac{2}{3}\right)^{9} = \frac{-512}{2187}$

Ex 8: Write the first six terms of the geometric sequence with $a_1 = 6$, $r = -\frac{1}{4}$.

$6, \frac{-3}{2}, \frac{3}{8}, \frac{-3}{32}, \frac{3}{128}, \frac{-3}{512}, \ldots$

Ex 9: If $a_2 = 3$ and $a_5 = \frac{3}{64}$ and $\{a_n\}$ is geometric, find a_1, a_7 and a formula for a_n.

$\underline{12}, 3, \underline{\quad}, \underline{\quad}, \frac{3}{64}, \underline{\quad}, \underline{\quad}, \ldots$

$\times\frac{1}{4}$ $\times r$ $\times r$ $\times r$

$3(r^3) = \frac{3}{64}$

$r^3 = \frac{1}{64}$

$r = \frac{1}{4}$

$a_1\left(\frac{1}{4}\right) = 3$

$\boxed{a_1 = 12}$

$\boxed{a_n = 12\left(\frac{1}{4}\right)^{n-1}}$

$a_7 = 12\left(\frac{1}{4}\right)^{6}$

$= \frac{12}{4^6} = \frac{3}{4^5}$

$\boxed{a_7 = \frac{3}{1024}}$

Ex 10: Identify each sequence from example 1 as geometric, arithmetic or neither and state a reason.

a) 2, 4, 8, ... geometric (r=2)

$2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8$

b) $\frac{2}{9}, \frac{3}{8}, \frac{4}{7}, \frac{5}{6}, \ldots$ no common difference between terms and no common ratio ⇒ neither

c) 2, 4, 16, ... neither

d) 19, 11, 3, -5, -13, ... arithmetic (d= -8)

$19 \xrightarrow{-8} 11 \xrightarrow{-8} 3 \xrightarrow{-8} -5 \xrightarrow{-8} -13$

e) 16, -8, 4, -2, ... geometric $(r = -\frac{1}{2})$

$16 \xrightarrow{\times -\frac{1}{2}} -8 \xrightarrow{\times -\frac{1}{2}} 4$

f) 1, 1, 2, 3, 5, 8, ... (Fibonacci) neither

g) 27, 18, 12, 8, ... geometric $(r = \frac{2}{3})$

$27 \xrightarrow{\times 2/3} 18 \xrightarrow{\times 2/3} 12 \xrightarrow{\times 2/3} 8$