2.4 ~ Sum and difference formulas

- Develop and use sum and difference formulas.
- Evaluate trigonometric functions using these formulas.
- Verify identities using these formulas.
- Solve more trigonometric equations.

Which are true and which are false?

1. $5(c+d) = 5c + 5d$ true (distributive property of multiplication through addition)

2. $\dfrac{a+b}{2} = \dfrac{a}{2} + \dfrac{b}{2}$

true

3. $(x+y)^2 = x^2 + y^2$ false (exponents do NOT distribute through addition) or subtraction

ex $(1+2)^2 = 3^2 = 9$

$1^2 + 2^2 = 1 + 4 = 5$ $\quad 9 \neq 5$

4. $\sqrt{p+q} = \sqrt{p} + \sqrt{q}$ false (roots do NOT distribute through addition or subtraction)

ex $\sqrt{4+9} = \sqrt{13}$

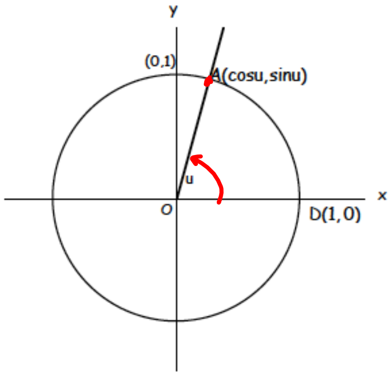
$\sqrt{4} + \sqrt{9} = 2 + 3 = 5$ $\quad 5 \neq \sqrt{13}$

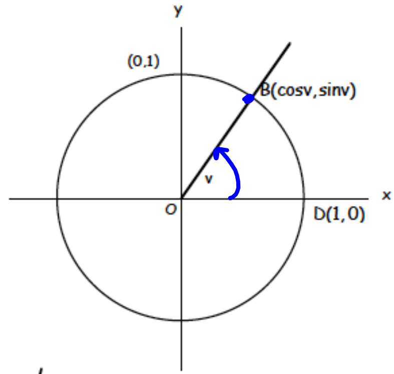
5. $\sin(u+v) = \sin u + \sin v$ false

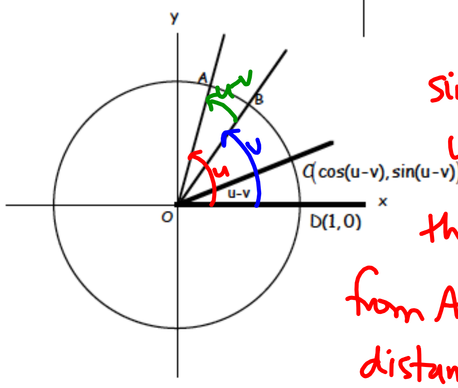
(sine does not distribute through addition or subtraction)

ex $\sin\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) = \sin(\pi) = 0$

$\sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$ $\quad 0 \neq \sqrt{3}$







Since angle u-v = angle u-v, the distance from A to B = distance from C to D

Distance from A to B

Distance from C to D

$$\left(\sqrt{(\cos u - \cos v)^2 + (\sin u - \sin v)^2}\right)^2 = \left(\sqrt{(\cos (u-v) - 1)^2 + (\sin (u-v) - 0)^2}\right)^2$$

difference of x-values

$$(\cos u - \cos v)(\cos u - \cos v) + (\sin u - \sin v)(\sin u - \sin v)$$
$$= (\cos(u-v) - 1)(\cos(u-v) - 1) + \sin^2(u-v)$$

$$(\cos^2 u - 2\cos u \cos v + \cos^2 v) + (\sin^2 u - 2\sin u \sin v + \sin^2 v) = \cos^2(u-v) - 2\cos(u-v) + 1 + \sin^2(u-v)$$

$$(\cos^2 u + \sin^2 u) + (\cos^2 v + \sin^2 v) - 2\cos u \cos v - 2\sin u \sin v = (\cos^2(u-v) + \sin^2(u-v)) - 2\cos(u-v) + 1$$

$$2 - 2\cos u \cos v - 2\sin u \sin v = 2 - 2\cos(u-v)$$

$$\cancel{1} - \cos u \cos v - \sin u \sin v = \cancel{1} - \cos(u-v)$$

$$\boxed{\cos u \cos v + \sin u \sin v = \cos(u-v)}$$

Simplifying the equation on the previous page gives us one of our sum/difference formulas for trigonometric expressions.

$$\cos (u - v) = \cos u \cos v + \sin u \sin v$$

In a similar way, we can derive these formulas: Sum/Difference Identities

$$\sin (u + v) = \sin u \cos v + \cos u \sin v$$

$$\sin (u - v) = \sin u \cos v - \cos u \sin v$$

$$\cos (u + v) = \cos u \cos v - \sin u \sin v$$

$$\cos (u - v) = \cos u \cos v + \sin u \sin v$$

These are the formulas for the sum/difference of a tangent.

$$\tan (u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\tan (u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Why do we need to know these?

Example 1:

a) Find the exact value of cos 75º .

cos 75º = cos (45º + 30º)

According to the formula this is:

$\cos(45^\circ+30^\circ) =$ cos 45º cos 30º - sin 45º sin 30º $= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$

$= \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$

b) $\sin\left(\frac{5\pi}{12}\right)$

$$\frac{5\pi}{12} = \frac{3\pi}{12} + \frac{2\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$$

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

Example 2: Verify this identity.

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x$$

$$= 1 \cdot \cos x - 0 \cdot \sin x$$

$$= \cos x \quad \checkmark$$

Example 3:

Solve this equation for all x on the interval $[0,2\pi)$.

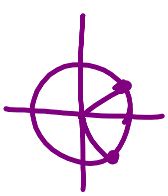
$$\sin\left(x+\frac{\pi}{4}\right)+\cos\left(x+\frac{\pi}{4}\right)=1$$

$$\sin x\cos\left(\frac{\pi}{4}\right)+\cos x\sin\left(\frac{\pi}{4}\right)+\cos x\cos\left(\frac{\pi}{4}\right)-\sin x\sin\left(\frac{\pi}{4}\right)=1$$

$$\frac{\sqrt{2}}{2}\left[\cancel{\sin x}+\cos x+\cos x-\cancel{\sin x}\right]=1$$

$$\frac{\sqrt{2}}{2}(2\cos x)=1$$

$$\sqrt{2}\cos x=1$$



$$\cos x=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)=\frac{\sqrt{2}}{2}$$

$$x=\frac{\pi}{4},\frac{7\pi}{4}$$

Example 4: Now one harder one:

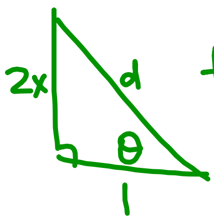
Simplify (state as an algebraic expression in terms of x).

sin (arctan(2x) + arccos x) $= \sin(\theta + \varphi)$

(assume φ & θ are both in Q1)

Let $\theta = \arctan(2x)$

$\varphi = \arccos(x)$



$\tan\theta = \dfrac{2x}{1}$

$d^2 = 1^2 + (2x)^2$

$d^2 = 1 + 4x^2$

$d = \sqrt{1+4x^2}$

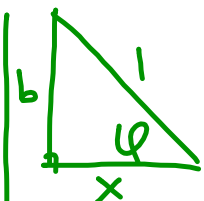
$\sin\theta = \dfrac{2x}{\sqrt{1+4x^2}}$

$\cos\theta = \dfrac{1}{\sqrt{1+4x^2}}$

$= \sin\theta\cos\varphi + \cos\theta\sin\varphi$

$= \left(\dfrac{2x}{\sqrt{1+4x^2}}\right)(x) + \left(\dfrac{1}{\sqrt{1+4x^2}}\right)\left(\sqrt{1-x^2}\right)$

$= \dfrac{2x^2 + \sqrt{1-x^2}}{\sqrt{1+4x^2}}$



$\cos\varphi = \dfrac{x}{1}$

$x^2 + b^2 = 1^2$

$b^2 = 1 - x^2$

$b = \sqrt{1-x^2}$

$\sin\varphi = \dfrac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$\cos\varphi = x$ (from above)