2.5 ~ Double Angle Formulas and Half-Angle Formulas

- Develop and use the double and half-angle formulas.
- Evaluate trigonometric functions using these formulas.
- Verify identities and solve more trigonometric equations.

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

sin(2u) = sin(u + u)

$= \sin u \cos u + \cos u \sin u$

$= \sin u \cos u + \sin u \cos u$

$\boxed{\sin(2u) = 2 \sin u \cos u}$ double angle identity for sine function

cos(2u) = cos(u + u)

①, ②, ③ are all double angle identities for cosine

$= \cos u \cos u - \sin u \sin u$

$\boxed{\cos(2u) = \cos^2 u - \sin^2 u}$ ①

$= (1-\sin^2 u) - \sin^2 u = \boxed{1 - 2\sin^2 u = \cos(2u)}$ ②

$\cos^2 u - \sin^2 u = \cos^2 u - (1-\cos^2 u) = \boxed{2\cos^2 u - 1 = \cos(2u)}$ ③

tan(2u) = tan(u + u)

$$= \frac{\tan u + \tan u}{1 - \tan u \tan u} = \boxed{\frac{2\tan u}{1-\tan^2 u} = \tan(2u)}$$

double angle identity for tangent

Why do we need these? Do they give us functions of new angles?

to solve equations, verify identities, and to do calculus later on.

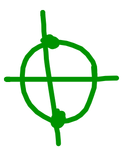
Example 1: Solve an equation with $2x$.

$$\sin(2x) + \cos x = 0$$

$$2 \sin x \cos x + \cos x = 0$$

$$\cos x \,(2 \sin x + 1) = 0$$

① $\cos x = 0$ OR ② $2 \sin x + 1 = 0$



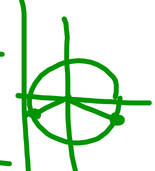
$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$$\boxed{x = \frac{(2n+1)\pi}{2}}$$

$n \in \mathbb{Z}$

$\sin x = -\frac{1}{2}$

$$\boxed{x = \begin{cases} -\frac{\pi}{6} + 2n\pi \\ \frac{7\pi}{6} + 2n\pi \end{cases}}$$



What about these?

$\sin(\frac{u}{2})$ $\cos(\frac{u}{2})$ $\tan(\frac{u}{2}) = \dfrac{\sin(\frac{u}{2})}{\cos(\frac{u}{2})}$

$\cos(2\theta) = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos(2\theta)$

$\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$

$\sin\theta = \pm\sqrt{\dfrac{1-\cos(2\theta)}{2}}$

Let $\theta = \frac{u}{2} \Rightarrow 2\theta = u$

$$\boxed{\sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos(u)}{2}}}$$

half-angle identity for sine

follow similar work

$$\boxed{\cos\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1+\cos(u)}{2}}}$$

half-angle identity for cosine

$$= \frac{\pm\sqrt{\frac{1-\cos u}{2}}}{\pm\sqrt{\frac{1+\cos u}{2}}}$$

$$= \frac{\pm\sqrt{1-\cos u}}{\pm\sqrt{1+\cos u}}$$

$$= \frac{\pm\sqrt{1-\cos u}}{\pm\sqrt{1+\cos u}}\left(\frac{\sqrt{1+\cos u}}{\sqrt{1+\cos u}}\right)$$

$$= \frac{\pm\sqrt{1-\cos^2 u}}{\pm(1+\cos u)}$$

$$= \frac{\pm\sqrt{\sin^2 u}}{\pm(1+\cos u)}$$

$$= \frac{\pm|\sin u|}{\pm(1+\cos u)}$$

$$\boxed{\tan\left(\frac{u}{2}\right) = \frac{\sin u}{1+\cos u}}$$

half-angle identity for tangent

<u>Important Note:</u>

For sine/cosine half-angle identities, you MUST decide whether you need + or − variety, depending on the angle.

Example 2: Use the formulas to compute the exact value of each of these.

a) $\sin 105^o$ $= \sin\left(\frac{1}{2}(210^\circ)\right)$

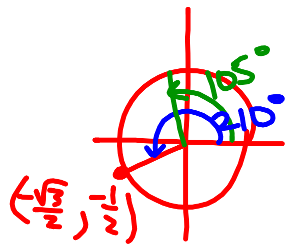
$$= \pm\sqrt{\frac{1-\cos(210^\circ)}{2}}$$

choose + variety because 105° in Q2

$$= \sqrt{\frac{1-\left(-\sqrt{3}/2\right)}{2}} = \sqrt{\frac{\frac{2}{2}+\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2+\sqrt{3}}{4}} = \boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}$$

(exact value)



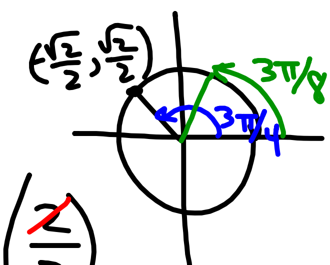
$$\sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos u}{2}}$$

b) $\tan \frac{3\pi}{8}$

$$= \tan\left(\frac{1}{2}\left(\frac{3\pi}{4}\right)\right)$$

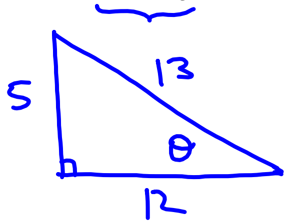
$$= \frac{\sin\left(\frac{3\pi}{4}\right)}{1+\cos\left(\frac{3\pi}{4}\right)} = \left(\frac{\frac{\sqrt{2}}{\cancel{2}}}{1+\frac{-\sqrt{2}}{2}}\right)\left(\frac{\cancel{2}}{2}\right)$$

$$= \boxed{\frac{\sqrt{2}}{2-\sqrt{2}}}$$



Example 3: Evaluate these expressions involving double or half angles.

If $\underbrace{\sin\theta = \frac{5}{13}}$, find $\sin(2\theta)$, $\cos\left(\frac{\theta}{2}\right)$ and $\tan(2\theta)$.



$$\sin(2\theta) = 2\sin\theta\cos\theta \quad \text{(double angle identity)}$$

$$= 2\left(\frac{5}{13}\right)\left(\frac{12}{13}\right)$$

$$= \frac{120}{169}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$= \pm\sqrt{\frac{1+\frac{12}{13}}{2}} = \pm\sqrt{\frac{13+12}{26}} = \pm\sqrt{\frac{25}{26}} = \frac{\pm 5}{\sqrt{26}}$$

we know $\sin\theta > 0$. This means θ is in either Q1 or Q2, i.e. $0 < \theta < \pi$

$$\Rightarrow \quad \tfrac{1}{2}(0) < \tfrac{1}{2}(\theta) < \tfrac{1}{2}\pi$$

$0 < \frac{\theta}{2} < \frac{\pi}{2}$, i.e. $\frac{\theta}{2}$ is in Q1.

$\Rightarrow \cos\left(\frac{\theta}{2}\right)$ must be positive

$$\Rightarrow \cos\left(\frac{\theta}{2}\right) = \frac{5}{\sqrt{26}}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{5}{12}\right)}{1-\left(\frac{5}{12}\right)^2} = \frac{\frac{5}{6}}{1-\frac{25}{144}}$$

$$= \frac{\frac{5}{6}}{\frac{144-25}{144}} = \frac{\frac{5}{6}}{\frac{119}{144}}$$

$$= \frac{5}{6}\cdot\frac{144}{119} = \frac{5(24)}{119} = \frac{120}{119}$$

Example 4:

Here is a problem you can work in two ways with very different results. Are they the same?

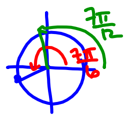
Find $\cos\left(\frac{7\pi}{12}\right)$.

a) using a half-angle formula:

$$\frac{7\pi}{12} = \frac{1}{2}\left(\frac{7\pi}{6}\right)$$

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\frac{7\pi}{6}}{2}\right)$$

$$= \pm\sqrt{\frac{1+\cos\left(\frac{7\pi}{6}\right)}{2}}$$



$$= \pm\sqrt{\left(\frac{1+{}^{-}\sqrt{3}/2}{2}\right)\left(\frac{2}{2}\right)}$$

$$= \pm\sqrt{\frac{2-\sqrt{3}}{4}}$$

$$= \pm\frac{\sqrt{2-\sqrt{3}}}{2}$$

$\frac{7\pi}{12}$ in Q2

$\Rightarrow \cos\left(\frac{7\pi}{12}\right)$ is negative.

$$= \boxed{\frac{-\sqrt{2-\sqrt{3}}}{2}}$$

b) using a sum/difference formula:

$$\frac{7\pi}{12} = \frac{3\pi}{12} + \frac{4\pi}{12}$$

$$= \frac{\pi}{4} + \frac{\pi}{3}$$

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$$

$$= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}$$

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let's first show

$$\left(\frac{-\sqrt{2-\sqrt{3}}}{2}\right)^2 = \left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^2$$

$$\left(\frac{-\sqrt{2-\sqrt{3}}}{2}\right)^2 = \frac{2-\sqrt{3}}{4}$$

$$\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^2 = \frac{(\sqrt{2}-\sqrt{6})(\sqrt{2}-\sqrt{6})}{4^2} = \frac{2-2\sqrt{12}+6}{16}$$

$$= \frac{8-2(2)\sqrt{3}}{16} = \frac{\cancel{4}(2-\sqrt{3})}{\cancel{16}_{\,4}} = \frac{2-\sqrt{3}}{4}$$

So $\left(\frac{-\sqrt{2-\sqrt{3}}}{2}\right)^2 = \left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^2$ <u>and</u>

both $\frac{-\sqrt{2-\sqrt{3}}}{2}$ and $\frac{\sqrt{2}-\sqrt{6}}{4}$ are negative

$$\Rightarrow \frac{-\sqrt{2-\sqrt{3}}}{2} = \frac{\sqrt{2}-\sqrt{6}}{4}.$$

Other formulas to be aware of:

Product-to-Sum Identities

$$\cos(a)\cos(b) = \frac{1}{2}\Big(\cos(a+b) + \cos(a-b)\Big)$$

$$\sin(a)\sin(b) = \frac{1}{2}\Big(\cos(a-b) - \cos(a+b)\Big)$$

$$\sin(a)\cos(b) = \frac{1}{2}\Big(\sin(a+b) + \sin(a-b)\Big)$$

$$\cos(a)\sin(b) = \frac{1}{2}\Big(\sin(a+b) - \sin(a-b)\Big)$$

Sum-to-Product Identities

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$