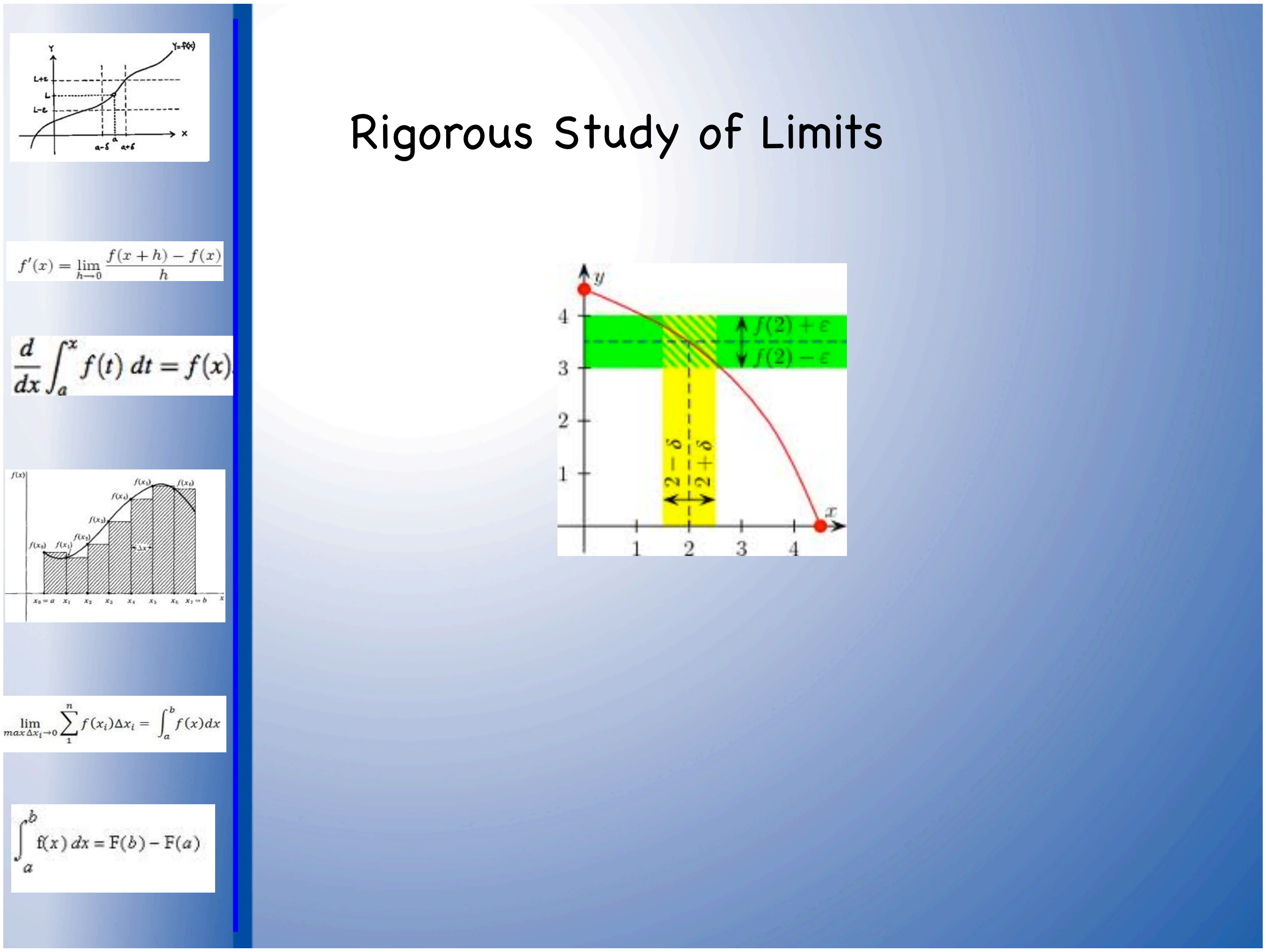
2.1B Riorous Study of Limits



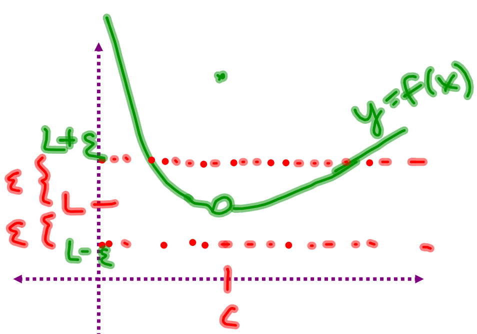
2.1B Riorous Study of Limits

Definition

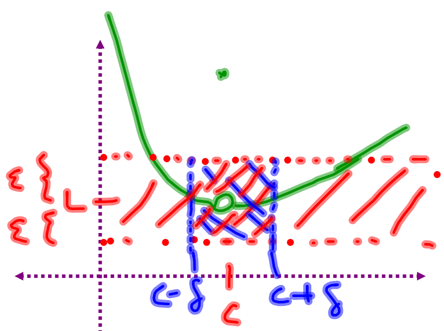
To say that $\lim_{x \to c} f(x) = L$ means that for every $\varepsilon > 0$ (no matter how small),

there exists a corresponding $\delta > 0$ such that $|f(x)-L|<\varepsilon$ provided that $0<|x-c|<\delta$;

that is, $0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon$



① choose an $\varepsilon > 0$.



② there is a $\delta > 0$ such that when $0<|x-c|<\delta$, then $|f(x)-L|<\varepsilon$

EX 1 Prove that $\lim_{x \to 3} (2x-5)=1$. $(c=3,\ L=1)$

scratch work

let $\varepsilon > 0$. we have to find a $\delta > 0$ s.t. (such that)

$0 < |x-3| < \delta$

$\Rightarrow |(2x-5)-1| < \varepsilon$

to make this true

need $|2x-6| < \varepsilon$

$2|x-3| < \varepsilon$

$|x-3| < \varepsilon/2$

$\Rightarrow$ choose $\delta = \varepsilon/2$.

Pf let $\varepsilon > 0$ be given.

Then choose $\delta = \varepsilon/2$.

$\Rightarrow$ if $0 < |x-3| < \delta = \varepsilon/2$,

then $2|x-3| < \varepsilon$

$|2x-6| < \varepsilon$

$|(2x-5)-1| < \varepsilon$

$\Rightarrow$ by defn

$\lim_{x \to 3} (2x-5) = 1$ #

(done with proof)

EX 2 Prove that $\lim_{x\to 1} \frac{2(x-1)(x+3)}{x-1} = 8$

scratchwork

$\left|\frac{2\cancel{(x-1)}(x+3)}{\cancel{(x-1)}} - 8\right| < \varepsilon$

$|2(x+3)-8| < \varepsilon$

$|2x+6-8| < \varepsilon$

$|2x-2| < \varepsilon$

$2|x-1| < \varepsilon$

$|x-1| < \varepsilon/2$

$\Rightarrow$ this tells me to choose $\delta = \varepsilon/2$.

Pf Fix $\varepsilon > 0$. Let $\delta = \varepsilon/2$.

Whenever $|x-1| < \delta = \frac{\varepsilon}{2}$

$\Leftrightarrow \quad 2|x-1| < \varepsilon$

$|2(x-1)| < \varepsilon$

$|2(x+3)-8| < \varepsilon$

$\left|\frac{2(x-1)(x+3)}{(x-1)} - 8\right| < \varepsilon$

$\Rightarrow$ by defn

$\lim_{x\to 1} \frac{2(x-1)(x+3)}{(x-1)} = 8$ #

EX 3 Prove that $\lim_{x \to c} \frac{1}{x-5} = \frac{1}{c-5}$ for all $c \neq 5$

scratch work

need to choose $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $|x - c| < \delta$.

i.e. want $\left|\frac{1}{x-5} - \frac{1}{c-5}\right| < \varepsilon$ whenever $|x-c| < \delta$.

$$\left|\frac{1}{x-5} - \frac{1}{c-5}\right| = \left|\frac{c-\cancel{5} - (x-\cancel{5})}{(x-5)(c-5)}\right| = \left|\frac{c-x}{(x-5)(c-5)}\right| = \left|\frac{x-c}{(x-5)(c-5)}\right| < \varepsilon$$

$$\iff |x-c|\left|\frac{1}{x-5}\right|\left|\frac{1}{c-5}\right| < \varepsilon \qquad (*)$$

♡start

note: $|c-5| = |c-5+x-x| = |(c-x)+(x-5)|$

(triangle inequality) $\leq |c-x| + |x-5|$

$\iff |c-5| - |x-c| \leq |x-5|$

$|x-5| \geq |c-5| - |x-c|$

$|x-5| > |c-5| - \delta$

$|x-c| < \delta \iff -|x-c| > -\delta$

choose $\delta \leq \frac{|c-5|}{2} \Rightarrow -\delta \geq -\frac{|c-5|}{2}$

$\rightarrow |x-5| > |c-5| - \frac{|c-5|}{2} = \frac{1}{2}|c-5|$

(♡) $\Rightarrow \boxed{\frac{1}{|x-5|} < \frac{2}{|c-5|}}$

$\Rightarrow (*)$ becomes

$$|x-c|\left|\frac{1}{x-5}\right|\left|\frac{1}{c-5}\right| \leq |x-c|\frac{2}{|c-5|}\left|\frac{1}{c-5}\right|$$

choose $\delta \leq \frac{\varepsilon|c-5|^2}{2}$

$\boxed{|x-c| < \delta}$

$$\Rightarrow |x-c|\left|\frac{1}{x-5}\right|\left|\frac{1}{c-5}\right| \leq \frac{\varepsilon|c-5|^2}{2}\left(\frac{2}{|c-5|^2}\right)$$

$$= \varepsilon$$

Pf Let $\varepsilon > 0$ be given.

choose $\delta = \min\left(\frac{|c-5|}{2}, \frac{\varepsilon |c-5|^2}{2} \right)$

Then $0 < |x-c| < \delta$,

$$\Rightarrow \left| \frac{1}{x-5} - \frac{1}{c-5} \right| = \left| \frac{c-x}{(x-5)(c-5)} \right|$$

$$= |x-c| \left| \frac{1}{x-5} \right| \left| \frac{1}{c-5} \right|$$

$$< \frac{\varepsilon |c-5|^{\cancel{2}}}{2} \left(\frac{1}{|x-5|} \right) \frac{1}{\cancel{|c-5|}}$$

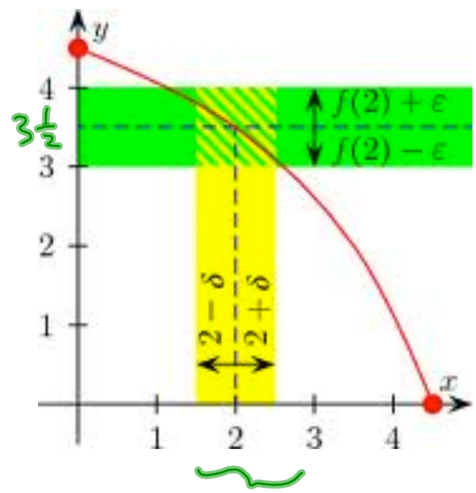
$$< \frac{\varepsilon |c-5|}{2} \left(\frac{2}{|c-5|} \right)$$

$$= \varepsilon$$

\#

(need ($\heartsuit$) from last page)

2.1B Riorous Study of Limits



$$\lim_{x \to 2} f(x) = 3\tfrac{1}{2}$$