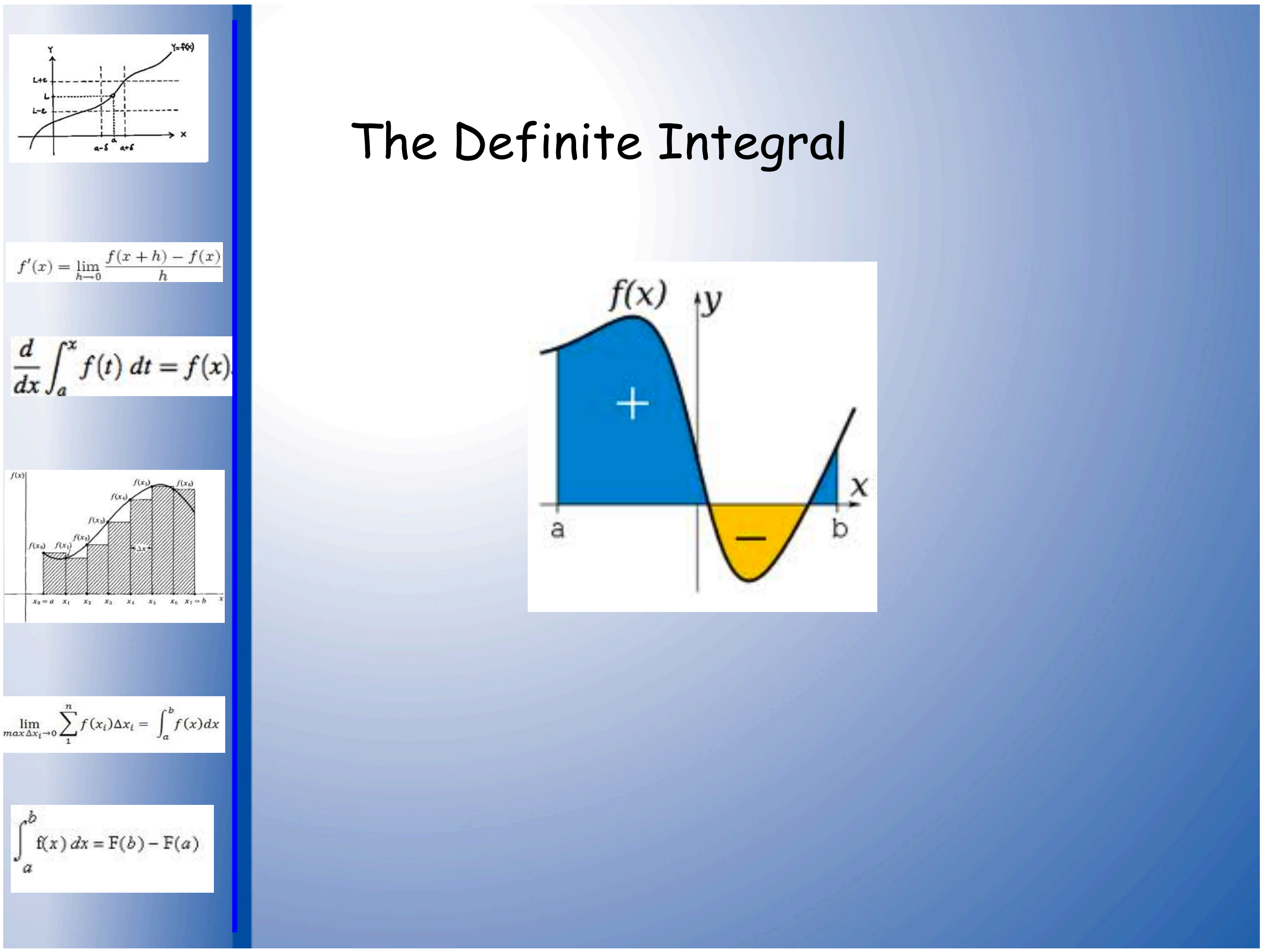
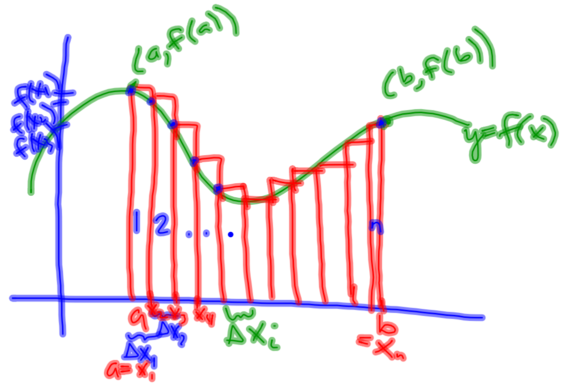

The Definite Integral
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$
$\lim_{max\,\Delta x_i \to 0} \sum_1^n f(x_i)\Delta x_i = \int_a^b f(x)dx$
$\int_a^b f(x)\,dx = F(b) - F(a)$
f(x)
y
x
a
b
+
–

<u>**The Definite Integral**</u>

want to calculate the area under a curve $y = f(x)$ from $x = a$ to $x = b$.

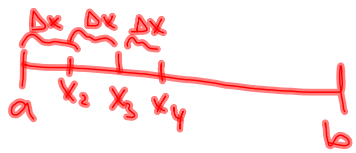


approx.

$$A = \underbrace{f(x_1)\Delta x_1}_{\text{area of rect 1}} + \underbrace{f(x_2)\Delta x_2}_{\text{area of rect 2}} + \ldots + f(x_n)\Delta x_n$$

since we can choose, let's let all $\Delta x_i = \Delta x$
(all rectangles have same width of Δx)

have n rectangles: $\boxed{\Delta x = \frac{b-a}{n}}$



$x_i = a + i\Delta x \quad i = 1, \ldots, n$

$\boxed{x_i = a + i\left(\frac{b-a}{n}\right)}$

$x_1 = a,\ x_2 = a + \Delta x,\ x_3 = a + 2\Delta x, \ldots$

$$\Rightarrow A = \sum_{i=1}^{n} f(x_i)\Delta x_i = \underbrace{\sum_{i=1}^{n} f\left(a + i\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right)}_{\text{approximate area}}$$

note: i is summation variable

because we can choose the height of each rectangle (i.e. we can choose right endpt, left endpt or something in between), I chose right endpt height.

exact area

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right)}$$

Definite Integral

Definition of the Definite Integral

Let f be a function that is defined on [a,b]. If $\lim_{|P|\to 0}\sum_{i=1}^{n} f(\overline{x}_i)\Delta x_i$ exists,

we say f is integrable on [a,b] and $\int_a^b f(x)dx = \lim_{|P|\to 0}\sum_{i=1}^{n} f(\overline{x}_i)\Delta x_i$.

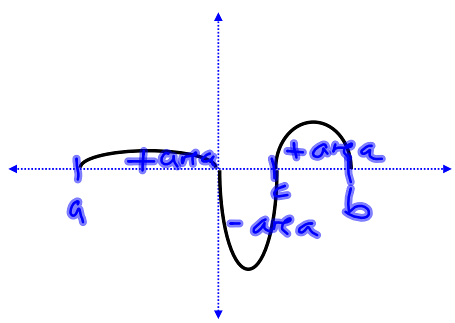
slightly different than my formula

Definite Integral

P = "partition"

(1) $\int_a^b f(x)dx = A_{up} - A_{down}$

signed area

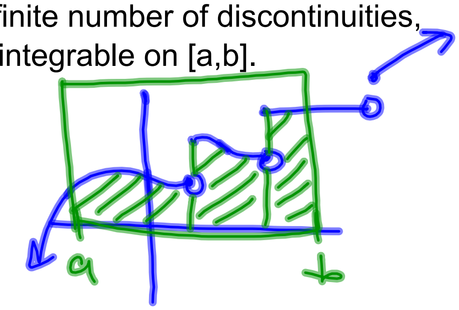


(2) $\int_a^a f(x)dx = 0$

(3) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

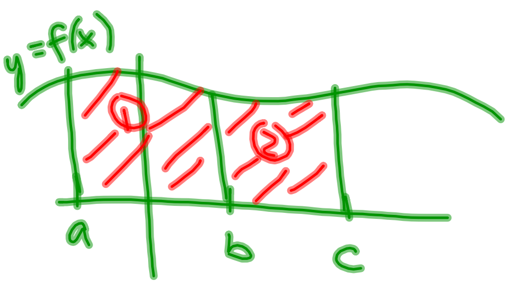
Integrability Theorem

If f is bounded on [a,b] and continuous there except for a finite number of discontinuities, then f is integrable on [a,b]. So, if f is continuous on [a,b] it is integrable on [a,b].



Interval Additive Property

If $f(x)$ is integrable, then $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.



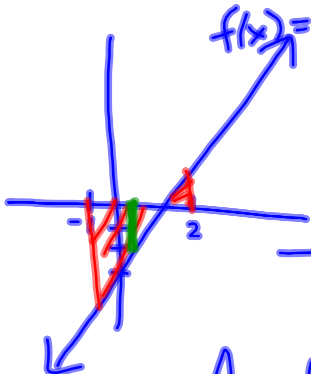
entire area
= area of (1) + area of (2)

EX 1 Evaluate this definite integral using the definition.

$b=2$, $a=-1$

$$A=\int_{-1}^{2}(2x-3)\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}f\left(a+i\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right)$$

$f(x)=y=2x-3$



$$=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\Delta x$$

$$\Delta x=\frac{b-a}{n}=\frac{2-(-1)}{n}=\frac{3}{n}$$

$$x_i=a+i\Delta x=-1+i\left(\frac{3}{n}\right)=-1+\frac{3i}{n}$$

$$A=\lim_{n\to\infty}\sum_{i=1}^{n}\underbrace{\left(2\left(-1+\frac{3i}{n}\right)-3\right)}_{f(x_i)}\underbrace{\left(\frac{3}{n}\right)}_{\Delta x}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(-2+\frac{6i}{n}-3\right)\frac{3}{n}=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{-15}{n}+\frac{18i}{n^2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{-15}{n}\sum_{i=1}^{n}1+\frac{18}{n^2}\sum_{i=1}^{n}i\right)$$

$$=\lim_{n\to\infty}\left(\frac{-15}{n}(n)+\frac{18}{n^2}\left(\frac{n(n+1)}{2}\right)\right)$$

$$=\lim_{n\to\infty}\left(-15+\frac{9n}{n}+\frac{9}{n}\right)$$

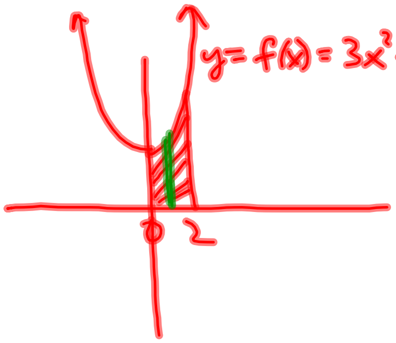
$$=\lim_{n\to\infty}\left(-6+\frac{9}{n}\right)=\boxed{-6}\text{ units}^2$$

($\frac{9}{n}\to 0$)

EX 2 Evaluate this definite integral using the definition.

$a=0$
$b=2$

$$A = \int_0^2 (3x^2+2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$



$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$$

$$\Rightarrow f(x_i) = 3\left(\frac{2i}{n}\right)^2 + 2 = \frac{12i^2}{n^2} + 2$$

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{12i^2}{n^2}+2\right)\frac{2}{n}$$

$$= \lim_{n\to\infty}\left(\frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{4}{n}\sum_{i=1}^{n} 1\right)$$

$$= \lim_{n\to\infty}\left(\frac{24}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{4}{n}(n)\right)$$

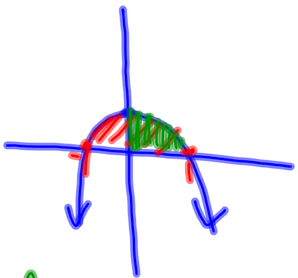
$$= \lim_{n\to\infty}\left(\frac{4}{n^2}(2n^2+3n+1) + 4\right)$$

$$= \lim_{n\to\infty}\left(8 + \frac{12}{n} + \frac{4}{n^2} + 4\right)$$

($\frac{12}{n} \to 0$, $\frac{4}{n^2} \to 0$)

$$= 8 + 4 = \boxed{12} \text{ units}^2$$

EX 3 Find the area of the region under the curve of $f(x) = -x^2 + 1$ on the interval [-1,1].
(To do this, divide the interval [-1,1] into n equal subintervals,
calculate the area of the circumscribed or inscribed rectangles
and take the limit as $n \to \infty$.)



A = 2·area of green region

$$A = \int_{-1}^{1} (-x^2+1)\,dx = 2\int_0^1 (-x^2+1)\,dx$$

$(-x^2+1)$: ht rect; dx: width rect

$a=0,\ b=1$

$$= 2\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_i = a + i\Delta x = 0 + i\left(\frac{1}{n}\right) = \frac{i}{n}$$

$$f(x_i) = -\left(\frac{i}{n}\right)^2 + 1 = \frac{-i^2}{n^2} + 1$$

$$= 2\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{-i^2}{n^2} + 1\right)\left(\frac{1}{n}\right)$$

$$= 2\lim_{n\to\infty} \left(\frac{-1}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n}\sum_{i=1}^{n} 1\right)$$

$$= 2\lim_{n\to\infty} \left(\frac{-1}{n^{3}}\left(\frac{n(2n+1)(n+1)}{6}\right) + \frac{1}{n}(n)\right)$$

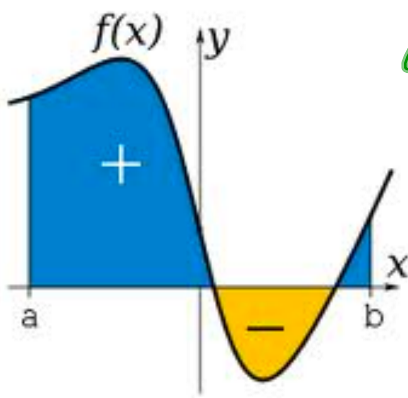
$$= 2\lim_{n\to\infty} \left(\frac{-1}{6n^2}(2n^2+3n+1) + 1\right)$$

$$= 2\lim_{n\to\infty} \left(\frac{-1}{3} + \frac{-1}{2n} - \frac{1}{6n^2} + 1\right)$$

($\frac{-1}{2n} \to 0$, $\frac{1}{6n^2} \to 0$)

$$= 2\left(\frac{-1}{3} + 1\right)$$

$$= 2\left(\frac{2}{3}\right) = \boxed{\frac{4}{3}} \text{ units}^2$$



area $= \int_a^b f(x)\,dx$