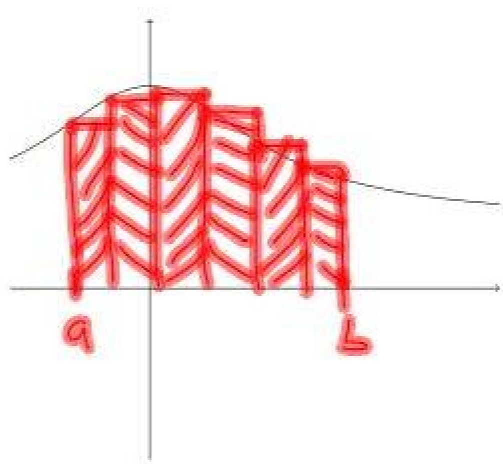
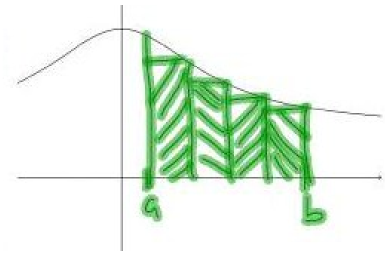
Math 1210 #34
Numerical Integration

If f(x) is continuous, we are guaranteed that $\int_a^b f(x)dx$ exists, but sometimes we cannot evaluate the integral. For these cases, we use numerical methods to approximate the definite integral (area under the curve.)

1. **Left Riemann Sum**



2. Right Riemann Sum



Area of n-th rectangle = $f(x_n)\Delta x_n$

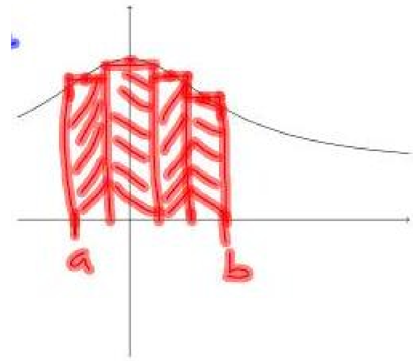
Let all $\Delta x_n = \Delta x$

$\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\left(\frac{b-a}{n}\right)\right)$$

Error: $E_n = -\frac{(b-a)^2}{2n} f'(c)$ for some $c \in [a, b]$

3. Midpoint Riemann Sum



Area of n-th rectangle = $f\left(\frac{x_{i-1}+x_i}{2}\right)\,\Delta x$

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

$$x_{i-1} = a + (i-1)\Delta x$$

$$\Rightarrow \frac{x_i x_{i-1}}{2} = \frac{a + i\Delta x + a + (i-1)\Delta x}{2}$$

$$= \frac{2a + 2i\Delta x - \Delta x}{2}$$

$$= a + i\Delta x - \frac{1}{2}\Delta x$$

4. Trapezoidal Rule

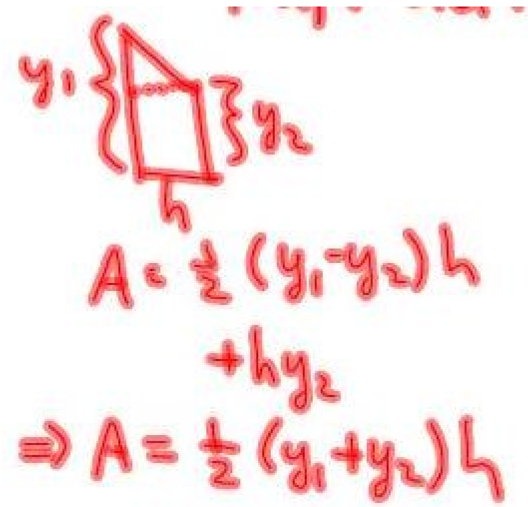
area of n^{th} trapezoid $= \frac{1}{2}\big(f(x_i) + f(x_{i-1})\big)\Delta x$

$$\Delta x = \frac{b-a}{n}, \qquad x_i = a + i\Delta x$$

$$x_{i+1} = a + (i-1)\Delta x$$

$$\int_a^b \mathrm{f}(x)\mathrm{dx} \approx \frac{1}{2}\left(\frac{b-a}{a}\right)\sum_{i=1}^{n}\left[f\left(x_{(-1)}\right) + f(x)\right]$$

Area of right trapezoid:



5. Simpon's Rule

(aka. Parabolic Rule)

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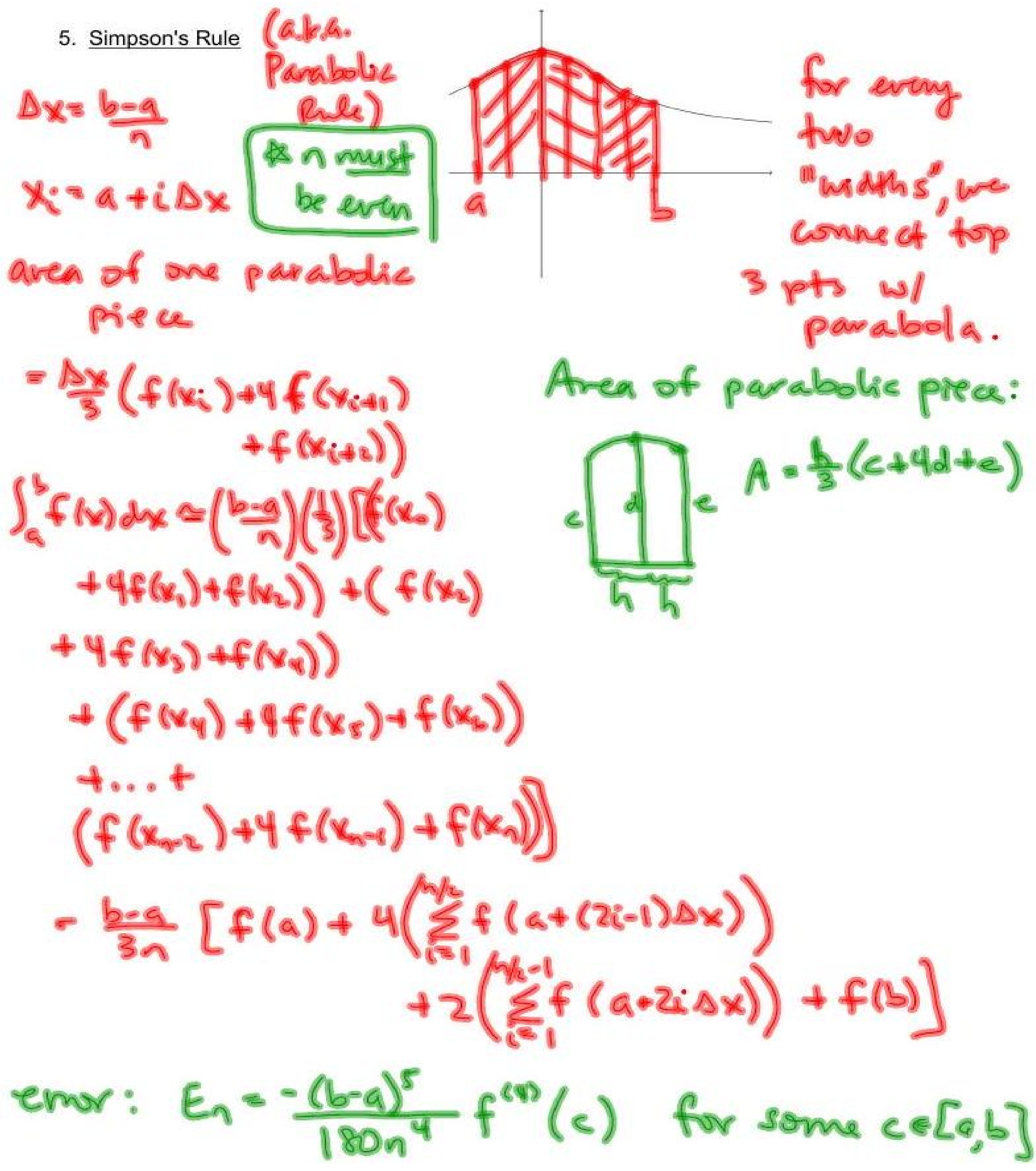


★ *n must be even*

for every two "widths", we connect top 3 pts w/ parabola.

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

area of one parabolic piece = $\frac{\Delta x}{3}\left(f(x_i) + 4f(x_{i+1}) + f(x_{i+2})\right)$

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{n}\right)\left(\frac{1}{3}\right)\begin{bmatrix}\left(f(x_0) + 4f(x_1) + f(x_2)\right) \\ +\left(f(x_2) + 4f(x_3) + f(x_4)\right) \\ +\left(f(x_4) + 4f(x_5) + f(x_6)\right) \\ +\ldots+\left(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)\end{bmatrix}$$

$$= \frac{b-a}{3n}\begin{bmatrix} f(a) + 4\left(\sum_{i=1}^{\frac{n}{2}} f\,(a + (2i-1)\Delta x)\right) \\ +2\left(\sum_{i=1}^{\frac{n}{2}-1} f\,(a + 2i\Delta x)\right) + f(b)\end{bmatrix}$$

error: $E_n = -\frac{(b-a)^5}{180n^4} f^{(4)}(c)$ for some $c \in [a, b]$

EX 1

Use methods 2, 4 and 5 to approximate this integral. $\int_1^3 \frac{1}{x^3}\,dx$ Let $n = 8$

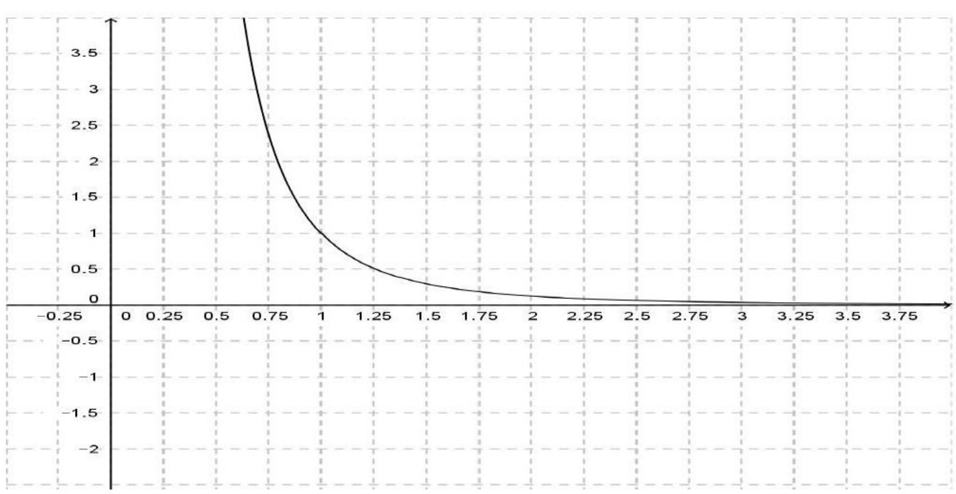
Right Rectangular Method

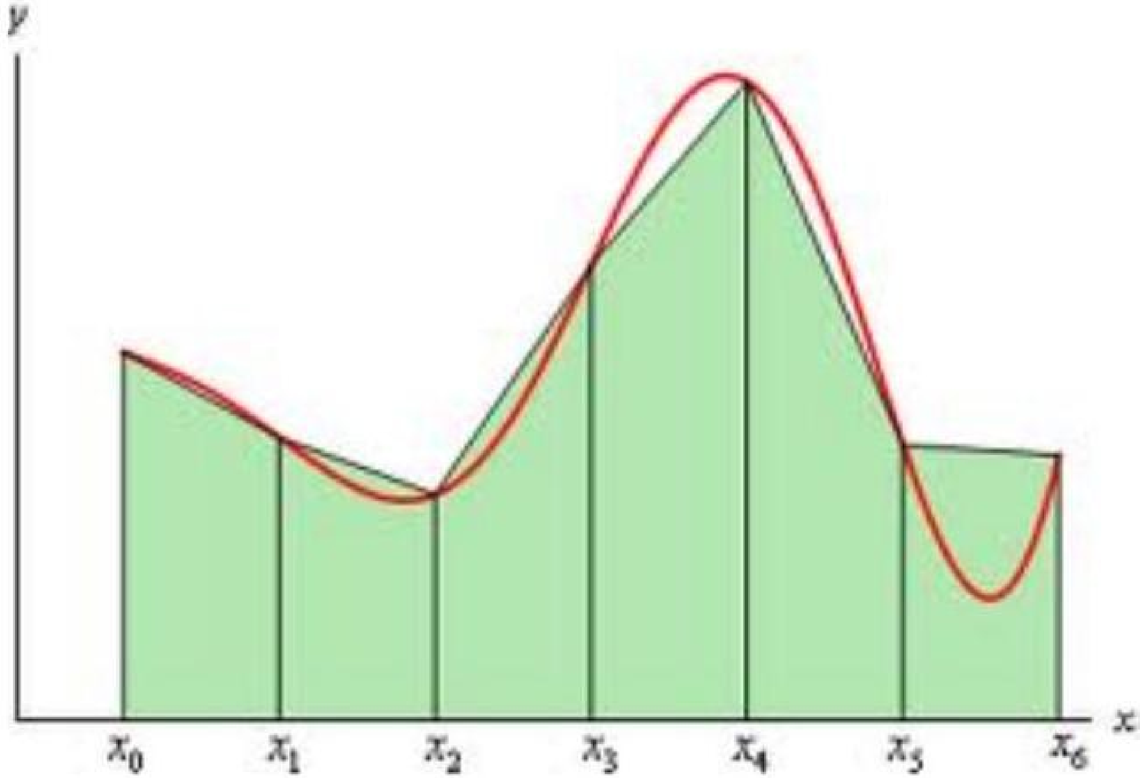
$\int_1^3 \frac{1}{x^3} dx$ Let $n = 8$.   Trapezoidal Rule

$\int_1^3 \frac{1}{x^3}\,dx$ Let n = 8. Simpson's Rule

Actual Value

$$\int_1^3 \frac{1}{x^3}\,dx$$




y
x
x_0
x_1
x_2
x_3
x_4
x_5
x_6