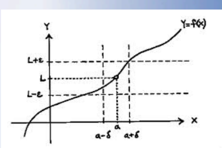
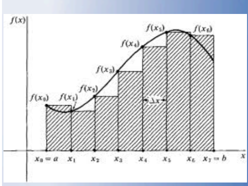


$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$



$$\lim_{max\,\Delta x_i \to 0} \sum_1^n f(x_i)\Delta x_i = \int_a^b f(x) dx$$

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

The Derivative

$$\begin{aligned} \frac{df}{dx} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \end{aligned}$$

9B Derivative

Definition: Derivative

The derivative of *f* is another function, *f'* such that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

provided the limit exists and is finite, for some x-value.

If *f'(c)* exists, we say *f(x)* is differentiable at *x* = *c*.

EX 1 Find *f'(x)* given $f(x) = 2\sqrt{x-1}, x \ge 1$

$$f'(x) = \lim_{h\to 0}\left(\frac{2\sqrt{x+h-1} - 2\sqrt{x-1}}{h}\right)\left(\frac{2\sqrt{x+h-1}+2\sqrt{x-1}}{2\sqrt{x+h-1}+2\sqrt{x-1}}\right)$$

$$= \lim_{h\to 0}\frac{4(x+h-1)+\cancel{4\sqrt{(x+h-1)(x-1)}}-\cancel{4\sqrt{(x+h-1)(x-1)}}-4(x-1)}{h\left(2\sqrt{x+h-1}+2\sqrt{x-1}\right)}$$

$$= \lim_{h\to 0}\frac{\cancel{4x}+4h-\cancel{4}-\cancel{4x}+\cancel{4}}{h\left(2\sqrt{x+h-1}+2\sqrt{x-1}\right)}$$

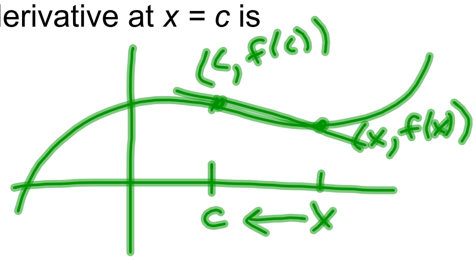
$$= \lim_{h\to 0}\frac{4\cancel{h}}{\cancel{h}\left(2\sqrt{x+h-1}+2\sqrt{x-1}\right)}$$

$$= \lim_{h\to 0}\frac{4}{2\sqrt{x+h-1}+2\sqrt{x-1}} = \frac{4}{2\sqrt{x-1}+2\sqrt{x-1}}$$

$$= \frac{4}{4\sqrt{x-1}} = \frac{1}{\sqrt{x-1}}$$

Another form of the definition of a derivative at $x = c$ is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$



EX 2 Use the above definition to find *h'(c)* if $h(x) = \dfrac{3}{x-5}$.

$$h'(c) = \lim_{x \to c} \frac{\frac{3}{x-5} - \frac{3}{c-5}}{x-c}$$

$$= \lim_{x \to c} \frac{1}{x-c}\left(\frac{3}{(x-5)}\left(\frac{c-5}{c-5}\right) - \frac{3}{(c-5)}\left(\frac{x-5}{x-5}\right)\right)$$

$$= \lim_{x \to c} \frac{1}{x-c}\left(\frac{3(c-5) - 3(x-5)}{(x-5)(c-5)}\right)$$

$$= \lim_{x \to c} \frac{3c - \cancel{15} - 3x + \cancel{15}}{(x-c)(x-5)(c-5)}$$

$x - c$
$= -(c - x)$

$$= \lim_{x \to c} \frac{-3\cancel{(x-c)}}{\cancel{(x-c)}(x-5)(c-5)}$$

$$= \lim_{x \to c} \frac{-3}{(x-5)(c-5)}$$

$$= \frac{-3}{(c-5)(c-5)}$$

$$= \boxed{\frac{-3}{(c-5)^2}}$$

EX 3 Each of these is a derivative for some function. Can you find the function?

a) $$f'(x)=\lim_{h\to 0}\frac{\overbrace{3(x+h)^2-2(x+h)}^{f(x+h)}-\overbrace{(3x^2-2x)}^{f(x)}}{h}$$

$f(x)=3x^2-2x$

b) $$f'(x)=\lim_{x\to 3}\frac{\overset{f(x)}{\frac{4}{x}}-\overset{f(3)}{\frac{4}{3}}}{x-3}=f'(3)$$

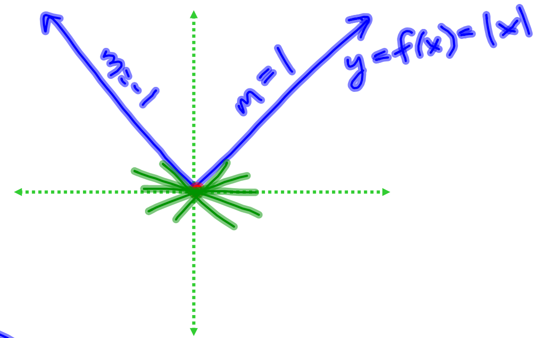
form

$$\left[f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right]$$

$f(x)=\frac{4}{x}$

EX 4 Let $f(x)=|x|$

Try to find $f'(0)$.



$$f'(x)=\lim_{h\to 0}\frac{|x+h|-|x|}{h}$$

$$=\lim_{h\to 0}\left(\frac{|x+h|-|x|}{h}\right)\left(\frac{|x+h|+|x|}{|x+h|+|x|}\right)$$

$$=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h(|x+h|+|x|)}$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h(|x+h|+|x|)}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}(|x+h|+|x|)}=\lim_{h\to 0}\frac{2x+h}{|x+h|+|x|}$$

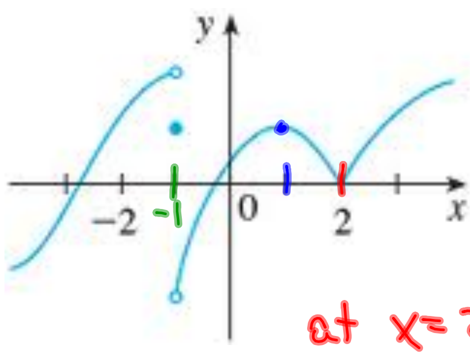
$$=\frac{2x}{|x|+|x|}=\frac{2x}{2|x|}=\frac{x}{|x|}$$

$$f'(x)=\frac{x}{|x|}=\begin{cases}1 & \text{if } x>0\\ \frac{x}{-x}=-1 & \text{if } x<0\\ \text{undefined} & \text{if } x=0\end{cases}$$

$\Rightarrow f(x)=|x|$ is <u>not</u> differentiable at $x=0$!!

$$f(x)=\begin{cases}x, & \text{if } x\geq 0\\ -x, & \text{if } x<0\end{cases} \Rightarrow f'(x)=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0\end{cases}$$

Visually, we can see a point where the derivative (slope of the curve) does not exist (DNE) by looking for "corners" or vertical tangents or "holes" in the graph of the function.



at x=-1, the fn is not continuous ⇒ the derivative DNE there

at x=1, the slope is zero

at x=2, the derivative DNE

What can we say about the derivative of this function at x = -1, 1 and 2?

Theorem: Differentiability and continuity

If *f'(x)* exists, then *f* is continuous at *x=c*.

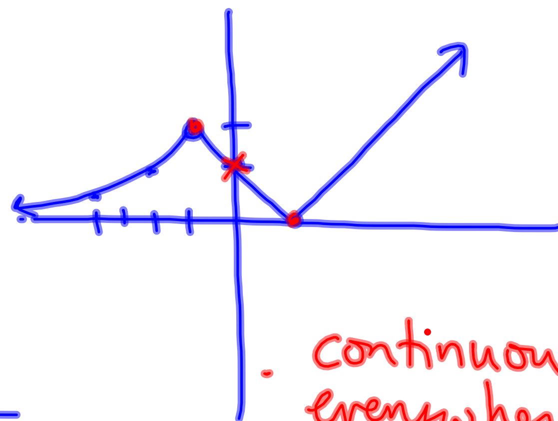
Also, if *f(x)* is discontinuous at x=c, then *f'(x)* does not exist.

(differentiability ⇒ continuity; differentiability is stronger)

Ex 5 Discuss the continuity and differentiability of this function at $x = -1, 0, 1$:

$$f(x) = \begin{cases} \frac{-2}{x} & \text{if } x < -1 \\ |x-1| & \text{if } x \geq -1 \end{cases}$$

x	y
-2	$\frac{-2}{-2} = 1$
-4	$\frac{-2}{-4} = \frac{1}{2}$
$\lim_{x \to -1^-} f(x)$	$\frac{-2}{-1} = 2$
-1	$\lvert -1-1 \rvert = 2$
0	$\lvert 0-1 \rvert = 1$



- continuous everywhere
- derivative is fine everywhere except at $x=-1$ and $x=1$

- at $x=0$, it's continuous and differentiable w/ slope of -1

$$
\begin{aligned}
\frac{df}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.
\end{aligned}
$$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$