If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

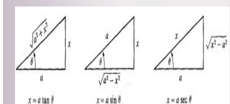
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

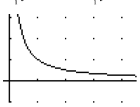
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

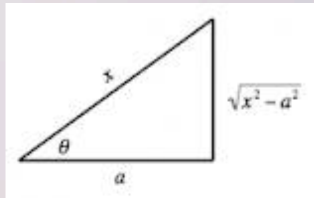
$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

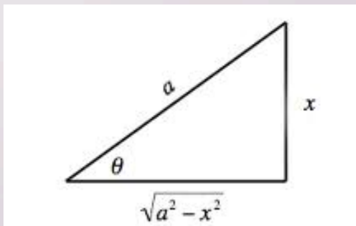
$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

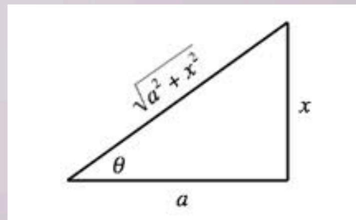
and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Rationalizing and Trigonometric Substitutions







Rationalizing Substitutions

Integrands involving $\sqrt[n]{ax+b}$

note: this is the n^{th} root of a linear polynomial

try $u=\sqrt[n]{ax+b}$

EX 1 $\displaystyle\int \frac{x^2+3x}{\sqrt{x+4}}dx$

$u=\sqrt{x+4}$

$\Rightarrow u^2=x+4$

$x=u^2-4$

$du=\frac{1}{2}(x+4)^{-1/2}dx$

$2\,du=\frac{1}{\sqrt{x+4}}dx$

$=2\int (u^2-4)^2+3(u^2-4)\,du$

$=2\int (u^4-8u^2+16+3u^2-12)\,du$

$=2\int (u^4-5u^2+4)\,du$

$=2\left(\frac{u^5}{5}-\frac{5u^3}{3}+4u\right)+C$

$=\boxed{\frac{2}{5}(x+4)^{5/2}-\frac{10}{3}(x+4)^{3/2}+8(x+4)^{1/2}+C}$

EX 2 $\displaystyle\int \frac{\sqrt{x}}{x+1}dx$

$u=\sqrt{x}$

$u^2=x$

$du=\frac{1}{2}x^{-1/2}dx$

$2\,du=\frac{1}{\sqrt{x}}dx$

$2\sqrt{x}\,du=dx$

$2u\,du=dx$

$=\int \frac{u}{u^2+1}(2u)\,du$

$=2\int \frac{u^2}{u^2+1}du$

$=2\int \frac{u^2+1-1}{u^2+1}du$

$=2\int \left(\frac{u^2+1}{u^2+1}-\frac{1}{u^2+1}\right)du$

$=2\int \left(1-\frac{1}{u^2+1}\right)du=2(u-\arctan u)+C$

$=2(\sqrt{x}-\arctan\sqrt{x})+C$

Integrals involving $\sqrt{a^2-x^2},\quad \sqrt{a^2+x^2},\quad \sqrt{x^2-a^2},\quad a\in\Re$

a) $\sqrt{a^2-x^2} \to let\ x=a\sin\theta \qquad \theta\in[-\pi/2,\pi/2]$

b) $\sqrt{a^2+x^2} \to let\ x=a\tan\theta \qquad \theta\in(-\pi/2,\pi/2)$

c) $\sqrt{x^2-a^2} \to let\ x=a\sec\theta \qquad \theta\in[0,\pi],\ \theta\neq\pi/2$

Note:

(a) $x=a\sin\theta$ then $\sqrt{a^2-x^2}$

$=\sqrt{a^2-a^2\sin^2\theta}$

$=\sqrt{a^2(1-\sin^2\theta)}$

$=\sqrt{a^2\cos^2\theta}$

$=a\cos\theta$

EX 3 $\displaystyle\int\frac{x^2}{\sqrt{16-x^2}}dx$

$16-x^2 \Rightarrow a=4$ (case (a))

let $x=4\sin\theta$

$\frac{dx}{d\theta}=4\cos\theta$

$dx=4\cos\theta\, d\theta$

$\sqrt{16-(4\sin\theta)^2}=\sqrt{16-16\sin^2\theta}$

$=\sqrt{16\cos^2\theta}=4\cos\theta$

$$=\int\frac{(4\sin\theta)^2}{\sqrt{16-(4\sin\theta)^2}}(4\cos\theta\, d\theta)$$

$$=\int\frac{16\sin^2\theta(\cancel{4\cos\theta})}{\cancel{4\cos\theta}}d\theta$$

$$=16\int\sin^2\theta\, d\theta$$

$$=\frac{16}{2}\int(1-\cos(2\theta))\,d\theta$$

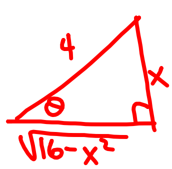
$$=8\left(\theta-\frac{\sin(2\theta)}{2}\right)+C=8\theta-4\sin(2\theta)+C$$

reminder
$x=4\sin\theta$
$\frac{x}{4}=\sin\theta$
$\sin^{-1}\left(\frac{x}{4}\right)=\theta$

$$=8\theta-4(2\sin\theta\cos\theta)+C$$

$$=8\theta-8\sin\theta\cos\theta+C$$

$$=8\sin^{-1}\left(\frac{x}{4}\right)-8\left(\frac{x}{4}\right)\cos\theta+C$$



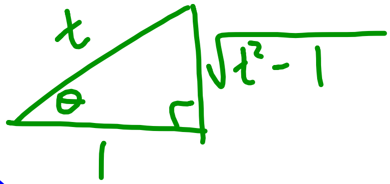
$\frac{x}{4}=\sin\theta$

$\Rightarrow\cos\theta=\frac{\sqrt{16-x^2}}{4}$

$$=8\sin^{-1}\left(\frac{x}{4}\right)-2x\left(\frac{\sqrt{16-x^2}}{4}\right)+C$$

$$=\boxed{8\sin^{-1}\left(\frac{x}{4}\right)-\frac{1}{2}x\sqrt{16-x^2}+C}$$

EX 4 $\displaystyle\int_2^3 \frac{dt}{t^2\sqrt{t^2-1}}$



$t = \sec\theta$

$dt = \sec\theta\tan\theta\, d\theta$

$\sqrt{t^2-1} = \dfrac{\sqrt{t^2-1}}{1} = \tan\theta$

$t=2,\quad 2=\sec\theta \iff \frac{1}{2}=\cos\theta$
$\theta = \pi/3$

$t=3,\quad 3=\sec\theta \iff \frac{1}{3}=\cos\theta \quad \theta=\cos^{-1}\left(\frac{1}{3}\right)$

$\theta = \sec^{-1}(3)$

$$= \int_{\pi/3}^{\sec^{-1}(3)} \frac{\cancel{\sec\theta}\ \cancel{\tan\theta}}{\sec^{\cancel{2}}\theta\ \cancel{\tan\theta}}\, d\theta$$

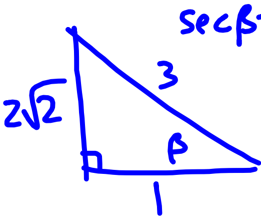
$$= \int_{\pi/3}^{\sec^{-1}(3)} \frac{1}{\sec\theta}\, d\theta$$

$$= \int_{\pi/3}^{\sec^{-1}(3)} \cos\theta\, d\theta$$

$$= \sin\theta\Big|_{\pi/3}^{\sec^{-1}(3)} = \sin\left(\sec^{-1}(3)\right) - \sin\left(\pi/3\right)$$

$\beta = \sec^{-1}(3)$

$\sec\beta = 3$



$\Rightarrow \sin\left(\sec^{-1}(3)\right)$

$= \sin\beta = \dfrac{2\sqrt{2}}{3}$

$$= \frac{2\sqrt{2}}{3} - \frac{\sqrt{3}}{2}$$

$$= \boxed{\frac{4\sqrt{2}-3\sqrt{3}}{6}}$$

Completing the Square (Use this strategy when there is a quadratic expression in the radical.)

EX 5 $\int \frac{3x}{\sqrt{x^2+4x-5}}dx$

x^2+4x-5

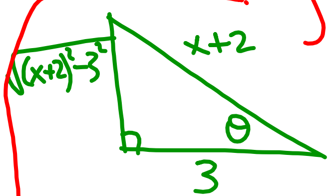
$=(x^2+4x+4)-5-4$

$=(x+2)^2-9$

$=(x+2)^2-3^2$

$$=\int \frac{3x}{\sqrt{(x+2)^2-3^2}}dx$$

$$=\int \frac{x}{\frac{\sqrt{(x+2)^2-3^2}}{3}}dx$$



have one leg $\sqrt{(x+2)^2-3^2}$

want $\sec\theta = \frac{x+2}{3}$

(don't want $\sin\theta$)

(" " $\tan\theta$)

$3\sec\theta = x+2$

$3\sec\theta - 2 = x$

$(3\sec\theta\tan\theta)d\theta = dx$

$$=\int \frac{(3\sec\theta-2)}{\cancel{\tan\theta}}(3\sec\theta\cancel{\tan\theta})d\theta$$

$$=3\int(3\sec\theta-2)\sec\theta\, d\theta$$

$$=3(3)\int \sec^2\theta\, d\theta - 6\int\sec\theta\, d\theta$$

$$=9\tan\theta - 6\ln|\sec\theta+\tan\theta|+c$$

$$=9\left(\frac{\sqrt{(x+2)^2-9}}{3}\right)-6\ln\left|\frac{x+2+\sqrt{(x+2)^2-9}}{3}\right|+c$$

$$=\boxed{3\sqrt{x^2+4x-5}-6\ln\left|x+2+\sqrt{x^2+4x-5}\right|+c}$$

note: $\ln\left|\frac{f(x)}{3}\right| = \ln|f(x)| - \underbrace{\ln 3}_{\text{just a constant}}$

(it can be "thrown in" with arbitrary constant c)

Conclusion

Two forms

① $\sqrt[n]{\text{linear polynomial}}$

try $u=\sqrt[n]{\text{linear polynomial}}$

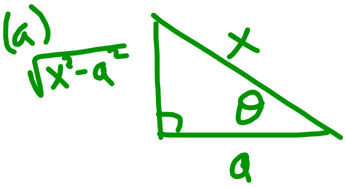
② Trig sub.

(a) $\sqrt{x^2-a^2}$

(b) $\sqrt{x^2+a^2}$

(c) $\sqrt{a^2-x^2}$

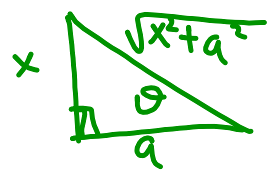
(a)



$\sec\theta = \frac{x}{a}$

$x = a\sec\theta$

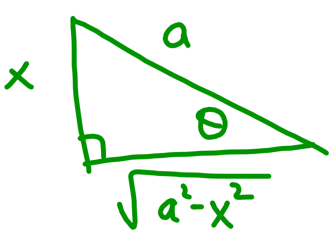
(b)



$\tan\theta = \frac{x}{a}$

$a\tan\theta = x$

(c)



$\sin\theta = \frac{x}{a} \iff x = a\sin\theta$