If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

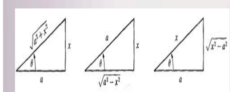
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

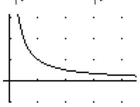
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x\frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Integration of Rational Functions Using Partial Fraction Decomposition

$$\begin{aligned} \frac{a}{x-2}+\frac{b}{(x+2)} &= \frac{4}{(x-2)(x+2)} \\ \frac{a(x+2)}{(x-2)(x+2)}+\frac{b(x-2)}{(x+2)(x-2)} &= \frac{4}{(x-2)(x+2)} \\ \frac{(a+b)x+2(a-b)}{(x-2)(x+2)} &= \frac{4}{(x-2)(x+2)} \end{aligned}$$

Partial Fraction Decomposition

A rational function is the quotient of two polynomials.

A proper rational function is the quotient of two polynomials where the numerator has a lower degree than the denominator.

improper rational fn: degree of numerator ≥ degree of denominator

Review of partial fraction decomposition (pfd)

ex $\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$: I have $\frac{3}{4}$.

EX 1 Rewrite this as a sum/difference of two fractions.

$$\frac{x-7}{x^2-x-12} = \frac{x-7}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3}$$

$$\cancel{(x-4)}\cancel{(x+3)}\left(\frac{x-7}{\cancel{(x-4)}\cancel{(x+3)}}\right) = \frac{A(x+3)\cancel{(x-4)}}{\cancel{x-4}} + \frac{B\cancel{(x+3)}(x-4)}{\cancel{x+3}}$$

Note: Before doing PFD, you must have a proper ratnl fn! So, if you have an improper rational fn, you must do long division first.

(*) $x-7 = A(x+3) + B(x-4)$

① if this eqn is true for all x-values, then I can plug in x-values I want.

$x=-3$: $-3-7 = 0A + -7B$

$-10 = -7B$

$\boxed{B = 10/7}$

$x=4$: $4-7 = 7A + 0B$

$-3 = 7A$

$\boxed{A = -3/7}$

$$\boxed{\frac{x-7}{(x-4)(x+3)} = \frac{-3/7}{x-4} + \frac{10/7}{x+3}}$$

② if (*) is true for all x-values, then we can equate coefficients of like terms, i.e. # of x's on left = # of x's on right, etc.

$x-7 = Ax + 3A + Bx - 4B$

$x-7 = x(A+B) + (3A-4B)$

① x: $1 = A+B$

② const: $-7 = 3A - 4B$

$A = 1-B \Longrightarrow -7 = 3(1-B) - 4B$

$-7 = 3 - 3B - 4B$

$-10 = -7B$

$B = 10/7$

$A = 1 - \frac{10}{7} = -\frac{3}{7}$

EX 2 $\displaystyle\int \frac{4x^2-6x+2}{x^2(x-1)(x+3)}\,dx$ (have a proper rational fn; do PFD)

$$\frac{4x^2-6x+2}{x^2(x-1)(x+3)} = \frac{Ax+B}{x^2} + \frac{C}{x-1} + \frac{D}{x+3}$$

$$\cancel{(x^2)(x-1)(x+3)}\frac{(4x^2-6x+2)}{x^2\cancel{(x-1)(x+3)}} = \frac{(Ax+B)\cancel{x^2}(x-1)(x+3)}{\cancel{x^2}} + \frac{Cx^2\cancel{(x-1)}(x+3)}{\cancel{x-1}} + \frac{Dx^2(x-1)\cancel{(x+3)}}{\cancel{x+3}}$$

$$4x^2-6x+2 = (Ax+B)(x-1)(x+3) + Cx^2(x+3) + Dx^2(x-1)$$

method 1

$x=1$: $4(1)-6(1)+2 = (A+B)(0) + C(4) + D(0)$

$0 = 4C \Rightarrow \boxed{C=0}$

$x=-3$: $4(9)-6(-3)+2 = 0+0+D(9)(-4)$

$56 = -36D$

$D = \frac{-56}{36} = \boxed{\frac{-14}{9} = D}$

$x=0$: $2 = B(-1)(3)$

$\boxed{B = -2/3}$

$x=-1$: $4(1)-6(-1)+2 = (-A+B)(-2)(2) + C(1)(2) + D(1)(-2)$

$12 = 4A - 4\left(\frac{-2}{3}\right) + 0 - 2\left(\frac{-14}{9}\right)$

$9(12) = \left(4A + \frac{8}{3} + \frac{28}{9}\right)9$

$108 = 36A + 24 + 28$

$-52 \qquad -52$

$56 = 36A$

$A = \frac{56}{36} = \boxed{\frac{14}{9}}$

$$\int \frac{4x^2-6x+2}{x^2(x-1)(x+3)}\,dx = \int \frac{\frac{14}{9}x + \frac{-2}{3}}{x^2}\,dx + \int \frac{-14/9}{x+3}\,dx$$

$$= \frac{14}{9}\int \frac{1}{x}\,dx + \frac{-2}{3}\int \frac{1}{x^2}\,dx - \frac{14}{9}\int \frac{1}{x+3}\,dx$$

$$= \frac{14}{9}\left(\ln|x|\right) - \frac{2}{3}\left(-1x^{-1}\right) - \frac{14}{9}\left(\ln|x+3|\right) + C$$

$$= \boxed{\frac{14}{9}\ln|x| + \frac{2}{3x} - \frac{14}{9}\ln|x+3| + C}$$

EX 3 $\int \frac{33x^2-7x+70}{(3x-2)(x^2+4)}dx$

degree of numerator = 2

degree of denominator = 3

$\Rightarrow$ this is proper rational fn $\Rightarrow$ do PFD!

$$\frac{33x^2-7x+70}{(3x-2)(x^2+4)} = \frac{A}{3x-2} + \frac{Bx+C}{x^2+4}$$

$$\cancel{(3x-2)}\cancel{(x^2+4)}\frac{(33x^2-7x+70)}{\cancel{(3x-2)}\cancel{(x^2+4)}} = \frac{A}{\cancel{(3x-2)}}\cancel{(3x-2)}(x^2+4) + \frac{(Bx+C)}{\cancel{(x^2+4)}}\cancel{(x^2+4)}(3x-2)$$

$$33x^2-7x+70 = A(x^2+4)+(Bx+C)(3x-2)$$

method (2) $\quad 33x^2-7x+70 = Ax^2+4A+3Bx^2-2Bx+3Cx-2C$

$$33x^2-7x+70 = x^2(A+3B)+x(-2B+3C)+(4A-2C)$$

x^2	x	const
① $33 = A+3B$	② $-7=-2B+3C$	③ $70=4A-2C$

$A=33-3B \longrightarrow 70=4(33-3B)-2C$

$70=132-12B-2C$

$-62=-12B-2C$

$\frac{12B-62}{-2}=\frac{-2C}{-2}$

② $-7=-2B+3(-6B+31) \longleftarrow -6B+31=C$

$-7=-2B-18B+93$

$-100=-20B \Rightarrow B=5 \longrightarrow C=-6(5)+31$

$\boxed{C=1}$

$A=33-3(5)=33-15=18=A$

$$\int\frac{33x^2-7x+70}{(3x-2)(x^2+4)}dx = \int\left(\frac{18}{3x-2}+\frac{5x+1}{x^2+4}\right)dx$$

$$=18\int\frac{1}{3x-2}dx+5\int\frac{x}{x^2+4}dx+\int\frac{1}{x^2+4}dx$$

$u=x^2+4$

$du=2x\,dx$

$\frac{1}{2}du=x\,dx$

$$=18\left(\frac{\ln|3x-2|}{3}\right)+5\int\frac{1/2}{u}du+\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C$$

$$=\boxed{6\ln|3x-2|+\frac{5}{2}\ln|x^2+4|+\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C}$$

EX 4 $\int \frac{\cos x}{\sin^4 x - 16}\,dx = \int \frac{1}{u^4-16}\,du$

$u = \sin x$

$du = \cos x\,dx$

$$= \int \frac{1}{(u^2-4)(u^2+4)}\,du$$

$$= \int \frac{1}{(u-2)(u+2)(u^2+4)}\,du$$

$$\frac{1}{(u-2)(u+2)(u^2+4)} = \frac{A}{u-2} + \frac{B}{u+2} + \frac{Cu+D}{u^2+4}$$

multiply both sides by $(u-2)(u+2)(u^2+4)$:

$$1 = A(u+2)(u^2+4) + B(u-2)(u^2+4) + (Cu+D)(u-2)(u+2)$$

method ①

$u=2$: $1 = A(4)(8) + 0B + 0C$

$\boxed{A = 1/32}$

$u=-2$: $1 = 0A + 0C + B(-4)(8)$

$\boxed{B = -1/32}$

$u=0$: $1 = A(2)(4) + B(-2)(4) + D(-4)$

$1 = \frac{1}{32}(8) - \frac{1}{32}(-8) - 4D$

$1 = \frac{1}{2} - 4D$

$\frac{-1}{4} \cdot \frac{1}{2} = -4D \cdot \frac{-1}{4}$

$\boxed{D = -1/8}$

$u=1$:

$1 = \frac{1}{32}(3)(5)$

$+ \frac{-1}{32}(-1)(5)$

$+ (C + \frac{-1}{8})(-3)$

$1 = \frac{15}{32} + \frac{5}{32} - 3C + \frac{3}{8}$

$1 = \frac{20}{32} + \frac{3}{8} - 3C$

$1 = 1 - 3C$

$-3C = 0$

$\boxed{C = 0}$

$$\int \frac{1}{(u-2)(u+2)(u^2+4)}\,du = \int \left(\frac{A}{u-2} + \frac{B}{u+2} + \frac{Cu+D}{u^2+4}\right) du$$

$$= \int \left(\frac{1/32}{u-2} - \frac{1/32}{u+2} + \frac{-1/8}{u^2+4}\right) du$$

$$= \frac{1}{32}\int \frac{1}{u-2}\,du - \frac{1}{32}\int \frac{1}{u+2}\,du - \frac{1}{8}\int \frac{1}{u^2+4}\,du$$

$$= \frac{1}{32}\ln|u-2| - \frac{1}{32}\ln|u+2| - \frac{1}{8}\left(\frac{1}{2}\right)\arctan\left(\frac{u}{2}\right) + C$$

$$= \boxed{\frac{1}{32}\ln|\sin x - 2| - \frac{1}{32}\ln|\sin x + 2| - \frac{1}{16}\arctan\left(\frac{\sin x}{2}\right) + C}$$

EX 5 $\displaystyle\int \frac{x^6-7x^4+11x^3-13x^2+x-6}{x^3-2x^2}\,dx$

$$= \int \frac{x^6-7x^4+11x^3-13x^2+x-6}{x^2(x-2)}\,dx$$

We <u>must</u> do long division first.

$$
\begin{array}{r}
x^3+2x^2-3x+5+\dfrac{-3x^2+x-6}{x^3-2x^2} \\
x^3-2x^2\,\overline{\big)\,x^6-7x^4+11x^3-13x^2+x-6} \\
-(x^6-2x^5) \\
\hline
2x^5-7x^4+11x^3 \\
-(2x^5-4x^4) \\
\hline
-3x^4+11x^3 \\
-(-3x^4+6x^3) \\
\hline
5x^3-13x^2+x-6 \\
-(5x^3-10x^2) \\
\hline
-3x^2+x-6
\end{array}
$$

$$\int \frac{x^6-7x^4+11x^3-13x^2+x-6}{x^3-2x^2}\,dx = \int\left[(x^3+2x^2-3x+5)+\frac{-3x^2+x-6}{x^3-2x^2}\right]dx$$

$$= \frac{x^4}{4}+\frac{2x^3}{3}-\frac{3x^2}{2}+5x+\underbrace{\int\frac{-3x^2+x-6}{x^2(x-2)}\,dx}_{\text{do PFD on this!}}$$

$$\int \frac{x^6-7x^4+11x^3-13x^2+x-6}{x^3-2x^2}\,dx$$

$$= \frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2} + 5x + \int \frac{-3x^2+x-6}{x^2(x-2)}\,dx$$

$$\frac{-3x^2+x-6}{x^2(x-2)} = \frac{Ax+B}{x^2} + \frac{C}{x-2}$$

$$\Rightarrow -3x^2+x-6 = (Ax+B)(x-2) + Cx^2$$

method ①

$x=0$: $\quad 0+0-6 = B(-2)$

$$\boxed{B=3}$$

$x=2$: $\quad -3(4)+2-6 = 0+4C$

$$-16 = 4C$$

$$\boxed{C=-4}$$

$x=1$:

$$-3+1-6 = (A+3)(-1) + -4$$

$$-8 = -A-3-4$$

$$-1 = -A$$

$$\boxed{A=1}$$

$$\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2} + 5x + \int \frac{-3x^2+x-6}{x^2(x-2)}\,dx$$

$$= \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2 + 5x + \int \left(\frac{x+3}{x^2} + \frac{-4}{x-2}\right)dx$$

$$= \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2 + 5x + \int \frac{1}{x}\,dx + \int \frac{3}{x^2}\,dx - 4\int \frac{1}{x-2}\,dx$$

$$= \boxed{\frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2 + 5x + \ln|x| + \frac{-3}{x} - 4\ln|x-2| + C}$$

Summary

- last resort integration technique.
- need rational fn (can only PFD on proper rational fn)