

if $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \frac{0}{0}$
or
 $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \frac{\infty}{\infty}$

Then $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)}$

provided that the latter limit exists.

$f(n) = f(n) + f(n)(b_n - a_n) + \frac{f'(n)(b_n - a_n)^2}{2!}$
 $+ \frac{f''(n)}{3!}(b_n - a_n)^3 + \frac{f'''(n)}{4!}(b_n - a_n)^4 + \dots$
 $+ \frac{f^{(k)}(n)}{k!}(b_n - a_n)^k$

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \dots$

$\int u dv = uv - \int v du$

Example: $\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$

Positive Series: Other Tests

Compare terms of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

Check the limit $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$

Check the ratio $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho$

Two types of series may converge.

Geometric Series: $\sum_{n=1}^{\infty} ar^n$ converges if $|r| < 1$.

p -series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

Ordinary Comparison Test

If $0 \leq a_n \leq b_n$ for every $n \geq N$

(i) If $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$.

EX 1 Does $\sum_{n=1}^{\infty} \frac{3n+4}{4n^2-2n-5}$ converge or diverge?

Limit Comparison Test

Assume $a_n \geq 0$ and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$.

If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ converge or diverge together.

If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

EX 2 Does this series converge or diverge?

$$\frac{1}{1^2+1} + \frac{2}{2^2+1} + \frac{3}{3^2+1} + \dots$$

EX 3 Does this series converge or diverge? $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2+1}$

Ratio Test

If $\sum a_n$ is a series of positive terms and $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho$,

then i) if $\rho < 1$, the series converges.

ii) if $\rho > 1$ or if $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \infty$, the series diverges.

iii) if $\rho = 1$, then the test is inconclusive.

EX 4 Does this series converge or diverge? $3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \dots$

EX 5 Does this series converge or diverge? $\sum_{n=1}^{\infty} \frac{n!}{5+n}$