If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

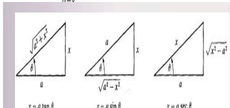
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

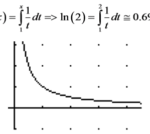
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots = \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$



$$\ln(x)=\int_1^x\frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv - \int v\,du$$

where it comes from:

the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Operations on Power Series

$$\frac{1}{1-x} = 1+x+x^2+x^3+\ldots \quad -1<x<1$$

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\ldots \quad -1<x<1$$

Operations on Power Series

Think of a power series as a polynomial with infinitely many terms.

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

Theorem A

Let $S(x) = \sum_{n=0}^{\infty} a_n x^n$ on the interval, I.

If x is interior to I, then

1) $S'(x) = \sum_{n=0}^{\infty} D_x\left(a_n x^n\right) = \sum_{n=0}^{\infty} n a_n x^{n-1}$

2) $\int_0^x S(t)dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

EX 1 We know

(assume n=0 case is 1)

$$1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad x \in (-1,1)$$

I (interval of convergence)

let's integrate.

$$\int_0^x \sum_{n=0}^{\infty} t^n \, dt = \sum_{n=0}^{\infty} \int_0^x t^n \, dt$$

$$= \sum_{n=0}^{\infty} \left(\frac{t^{n+1}}{n+1} \Big|_0^x \right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{x^{n+1}}{n+1} - 0 \right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{x^{n+1}}{n+1} \right)$$

Also $\int_0^x \frac{1}{1-t} dt = \frac{\ln|1-t|}{-1} \Big|_0^x$

$$= -\ln|1-x| - -\ln|1|$$

$$= -\ln|1-x|$$

$\Rightarrow$ Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ when $|x|<1$

then $\int_0^x \frac{1}{1-t} dt = \int_0^x \sum_{n=0}^{\infty} t^n \, dt$

$$-\ln|1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

$$\boxed{-\ln|1-x| = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots, \quad |x|<1}$$

EX 2 Show $S'(x)=S(x)$ for $S(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\ldots$.

($n=0$, $n=1$, $n=2$, $n=3$, $n=4$)

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

You must first demonstrate convergence, then solve $S'(x)=S(x)$.

Notice $S(0) = 1$.

show convergence (use ART to find conv. set)

ART: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$

$= |x| \lim_{n\to\infty}\left|\frac{n!}{(n+1)n!}\right| = |x|(0) = 0 < 1$

$\Rightarrow$ the convergence set for this power series is $\mathbb{R}$ (or $(-\infty,\infty)$)

claim $S(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!} \Rightarrow S'(x)=S(x)$.

Pf $S'(x) = D_x\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)$

$= D_x\left(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)$

$= 1+x+\frac{3x^2}{3!}+\frac{4x^3}{4!}+\ldots$

$= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$

$= S(x)$ //

$\Rightarrow$ convergence set for $S'(x)$ is $\mathbb{R}$

notice: $S(0) = 1+0+\frac{0^2}{2!}+\frac{0^3}{3!}+\ldots = 1$

what fn do we know of that meets these conditions? ① $S(0)=1$

② $S'(x)=S(x)$

$\Rightarrow S(x) = \boxed{e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}}$ for all $x\in\mathbb{R}$.

EX 3 Find the power series for $f(x)=\dfrac{x}{1+x^2}$.

know: $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$

$$\frac{1}{1-\heartsuit} = \sum_{n=0}^{\infty} \heartsuit^n, \quad |\heartsuit|<1$$

$$\Rightarrow \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n \qquad |-x^2|<1$$

$$(\heartsuit=-x^2) \quad = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad \Leftrightarrow \quad |x|<1$$

$$\Rightarrow f(x)=\frac{x}{1+x^2} = x\left(\frac{1}{1+x^2}\right) = x\sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}, \quad |x|<1}$$

Theorem B

If $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$ with both series converging

for $|x|<r$, we can perform arithmetic operations and the resulting series will converge for $|x|<r$. (If $b_0 \neq 0$, the result holds for division, but we can guarantee its validity only for $|x|$ sufficiently small.)

EX 4 Find a power series for $f(x) = sinh(x)$.

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left(e^x - e^{-x}\right)$$

know

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R} \qquad \text{so} \qquad e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}, \quad x \in \mathbb{R}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$\Rightarrow e^x - e^{-x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right)$$

$$= x - (-x) + \frac{x^3}{3!} - \left(\frac{-x^3}{3!}\right) + \frac{x^5}{5!} - \left(\frac{-x^5}{5!}\right) + \cdots$$

$$= 2x + 2\left(\frac{x^3}{3!}\right) + 2\left(\frac{x^5}{5!}\right) + \cdots$$

$$\Rightarrow \frac{1}{2}\left(e^x - e^{-x}\right) = \frac{1}{2}\left[2x + 2\left(\frac{x^3}{3!}\right) + 2\left(\frac{x^5}{5!}\right) + \cdots\right]$$

$$= \underset{n=0}{x} + \underset{n=1}{\frac{x^3}{3!}} + \underset{n=2}{\frac{x^5}{5!}} + \underset{n=3}{\frac{x^7}{7!}} + \cdots$$

$$\boxed{\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R}}$$

EX 5 Find the power series for $f(x) = \dfrac{\arctan(x)}{1+x^2+x^4}$.

(given: $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, $|x| < 1$)

$$f(x) = \frac{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots}{1+x^2+x^4}$$

do long division!

$x - \frac{4}{3}x^3 + \frac{8}{15}x^5 + \dots$ $|x| < 1$

$1 + x^2 + x^4 \overline{\big) \, x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots}$

$-(x + x^3 + x^5)$

$-\frac{4}{3}x^3 - \frac{4}{5}x^5 - \frac{x^7}{7}$

$-(-\frac{4}{3}x^3 - \frac{4}{3}x^5 - \frac{4}{3}x^7)$

$\frac{8}{15}x^5 + \frac{25}{21}x^7$

$\frac{4}{3} - \frac{4}{5} = \frac{8}{15}$

$\frac{4}{3} - \frac{1}{7} = \frac{25}{21}$

EX 6 Find these sums.

a) $\underset{n=0}{1} + \underset{n=1}{x^2} + \underset{n=2}{x^4} + \underset{n=3}{x^6} + \underset{n=4}{x^8} + \ldots = \sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} (x^2)^n$

Know: $\frac{1}{1-\heartsuit} = \sum_{n=0}^{\infty} \heartsuit^n, \; |\heartsuit| < 1$

$= \boxed{\frac{1}{1-x^2}, \; |x|<1}$

b) $\underset{n=1}{\cos x} + \underset{n=2}{\cos^2 x} + \underset{n=3}{\cos^3 x} + \underset{n=4}{\cos^4 x} + \ldots$

$= \sum_{n=1}^{\infty} (\cos x)^n = \sum_{n=0}^{\infty} (\cos x)^n - 1$

$= \frac{1}{1-\cos x} - 1, \quad |\cos x| < 1$

$\cos x + \cos^2 x + \cos^3 x + \cos^4 x + \ldots$

$= \frac{1}{1-\cos x} - 1$ whenever $|\cos x| < 1$

$\iff x \in \mathbb{R}, \; x \neq n\pi \quad n \in \mathbb{Z}$

Conclusion

- we can arithmetically manipulate power series we know to create new power series (w/ same convergence set)
- we can differentiate or integrate power series term-wise; we get new power series (w/ same convergence set)