

If  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$   
 or  
 $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$   
 Then  
 $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$   
 provided that the latter limit exists.

$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$   
 $= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ .

$\ln(x) = \int_1^x \frac{1}{t} dt \Rightarrow \ln(2) = \int_1^2 \frac{1}{t} dt \approx 0.69315$

$\int u dv = uv - \int v du$

where it comes from:  
 $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$   
 put into reverse:  
 $uv' = \frac{d}{dx}(uv) - v \frac{du}{dx}$   
 and then rearrange:  
 $\int uv' dx = uv - \int v \frac{du}{dx} dx$

# Operations on Power Series

$$\frac{1}{1-x} = 1+x+x^2+x^3+\dots \quad -1 < x < 1$$

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\dots \quad -1 < x < 1$$

## Operations on Power Series

Think of a power series as a polynomial with infinitely many terms.

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

### Theorem A

Let  $S(x) = \sum_{n=0}^{\infty} a_n x^n$  on the interval, I.

If x is interior to I, then

- 1)  $S'(x) = \sum_{n=0}^{\infty} D_x(a_n x^n) = \sum_{n=0}^{\infty} n a_n x^{n-1}$
- 2)  $\int_0^x S(t) dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

EX 1 We know

$$1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad x \in (-1, 1)$$

EX 2 Show  $S'(x)=S(x)$  for  $S(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$  .

You must first demonstrate convergence, then solve  $S'(x)=S(x)$ .

Notice  $S(0) = 1$ .

EX 3 Find the power series for  $f(x) = \frac{x}{1+x^2}$  .

Theorem B

If  $f(x) = \sum a_n x^n$  and  $g(x) = \sum b_n x^n$  with both series converging

for  $|x| < r$ , we can perform arithmetic operations and the resulting series will converge for  $|x| < r$ . (If  $b_0 \neq 0$ , the result holds for division, but we can guarantee its validity only for  $|x|$  sufficiently small.)

EX 4 Find a power series for  $f(x) = \sinh(x)$ .

EX 5 Find the power series for  $f(x) = \frac{\arctan(x)}{1+x^2+x^4}$ .

EX 6 Find these sums.

a)  $1 + x^2 + x^4 + x^6 + x^8 + \dots$

b)  $\cos x + \cos^2 x + \cos^3 x + \cos^4 x + \dots$