Taylor and Maclaurin Series

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^{n}$$

If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

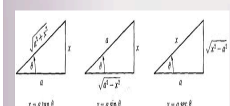
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

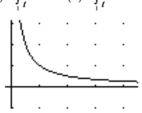
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x\frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt\cong 0.69315$$



$$\int u\,dv=uv-\int v\,du$$

where it comes from:

the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

$$\int\frac{d}{dx}(uv)=\int(u\frac{dv}{dx}+v\frac{du}{dx})$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Taylor and Maclaurin Series

If we represent some function f(x) as a power series in (x-a), then

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \ldots$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \ldots$$

$$f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \ldots$$

$$f'''(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + \ldots$$

$$f^{(4)}(x) = 4\cdot 3\cdot 2c_4 + 5\cdot 4\cdot 3c_5(x-a) + \ldots$$

Let $x=a$. $f(a) = c_0$ $f''(a) = 2c_2$ $f^{(4)}(a) = 4\cdot 3\cdot 2c_4$

$f'(a) = c_1$ $f'''(a) = 3\cdot 2c_3$

In general, $f^{(n)}(a) = n!\,c_n$ $n = 0, 1, 2, \ldots$

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

Uniqueness Theorem

Suppose $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$

for every x in some interval around a.

Then $\boxed{c_n = \dfrac{f^{(n)}(a)}{n!}}$.

Taylor's Formula with Remainder

Let $f(x)$ be a function such that $f^{(n+1)}(x)$ exists for all x on an open interval containing a.

Then, for every x in the interval,

$$f(x)=f(a)+f'(a)\,(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots+\frac{f^{n}(a)}{n!}(x-a)^n+R_n(x)$$

Taylor's Formula

where $R_n(x)$ is the remainder (or error). $R_n(x)=\dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

Taylor's Theorem

Let f be a function with all derivatives in $(a-r,a+r)$.

The Taylor Series $f(a)+f'(a)\,(x-a)+\dfrac{f''(a)}{2!}(x-a)^2+\ldots$

represents $f(x)$ on $(a-r,a+r)$

if and only if $\lim_{n\to\infty} R_n(x)=0$, $\quad R_n(x)=\dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$.

$$c\in(a-r,a+r)$$

EX 1 Find the Maclaurin series for $f(x)=\cos x$ and prove it represents $\cos x$ for all x.

$f(x)=\cos x$ | $f(0)=\cos 0=1$

$f'(x)=-\sin x$ | $f'(0)=0$

$f''(x)=-\cos x$ | $f''(0)=-1$

$f'''(x)=\sin x$ | $f'''(0)=0$

$f^{(4)}(x)=\cos x$ | $f^{(4)}(0)=1$

$a=0$

Note:

Maclaurin Series = a Taylor series w/ $a=0$

$$\Rightarrow f(x)=f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\dots$$

$$=1+0+\frac{-1}{2}x^2+0+\frac{x^4}{4!}+0-\frac{1}{6!}x^6+\frac{1}{8!}x^8-\frac{1}{10!}x^{10}+\dots$$

$$=\underset{n=0}{1}-\underset{n=1}{\frac{1}{2}x^2}+\underset{n=2}{\frac{1}{4!}x^4}-\underset{n=3}{\frac{1}{6!}x^6}+\underset{n=4}{\frac{1}{8!}x^8}-\frac{1}{10!}x^{10}+\dots$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$ (Maclaurin series for $\cos x$)

We need to show $\lim_{n\to\infty} R_n(x)=0$.

(then we know this power series represents $\cos x$ $\forall x$)

(☆ do at end of this lecture)

EX 2 Find the Maclaurin series for $f(x) = \sin x$.

$a=0$

$f(x)=\sin x$	$f(a)=f(0)=0$
$f'(x)=\cos x$	$f'(0)=1$
$f''(x)=-\sin x$	$f''(0)=-0=0$
$f'''(x)=-\cos x$	$f'''(0)=-1$
$f^{(4)}(x)=\sin x$	$f^{(4)}(0)=0$

$$f(x)= \sin x = f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\dots$$

$$=\underset{n=0}{0+x}+0+\underset{n=1}{\frac{-1}{3!}x^3}+0+\underset{n=2}{\frac{1}{5!}x^5}+0+\underset{n=3}{\frac{-1}{7!}x^7}+\underset{n=4}{\frac{1}{9!}x^9}+\dots$$

$$\boxed{\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

this converges for all $x \in \mathbb{R}$

EX 3 Write the Taylor series for $f(x)=\dfrac{1}{x}$ centered at $a=1$.

$f(x)=\frac{1}{x}$ $\quad f(1)=1$

$f'(x)=\frac{-1}{x^2}$ $\quad f'(1)=-1$

$f''(x)=\frac{2}{x^3}$ $\quad f''(1)=2$

$f'''(x)=\frac{-6}{x^4}$ $\quad f'''(1)=-6$

$f^{(4)}(x)=\frac{4!}{x^5}$ $\quad f^{(4)}(1)=4!$

pattern suggests $f^{(n)}(1)=(-1)^n\, n!$

$$\Rightarrow f(x)=\frac{1}{x}=f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3+\dots$$

$$=1-(x-1)+\frac{2}{2!}(x-1)^2+\frac{-6}{3!}(x-1)^3+\frac{4!}{4!}(x-1)^4+\dots$$

$$+\frac{(-1)^n n!}{n!}(x-1)^n+\dots$$

$$=\underset{n=0}{1}-\underset{n=1}{(x-1)}+\underset{n=2}{(x-1)^2}-\underset{n=3}{(x-1)^3}+(x-1)^4+\dots$$

$$\boxed{\frac{1}{x}=\sum_{n=0}^{\infty}(-1)^n(x-1)^n}$$

know

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n,\quad |x|<1$$

we have $\frac{1}{x}=\frac{1}{1-(1-x)}$ of this form

$\Rightarrow$ interval of convergence

$|1-x|<1$

$|x-1|<1$

convergence set for $f(x)=\frac{1}{x}$, centered at $a=1$. $\boxed{0<x<2}$

EX 4 Find the Taylor series for $f(x) = \sin x$ in $(x-\pi/4)$.

$a = \pi/4$

$$f(x) = \sin x = f(\pi/4) + f'(\pi/4)(x-\pi/4) + \frac{f''(\pi/4)}{2!}(x-\pi/4)^2 + \frac{f'''(\pi/4)}{3!}(x-\pi/4)^3 + \cdots$$

$f(x) = \sin x$ $\quad f(\pi/4) = \sqrt{2}/2$

$f'(x) = \cos x$ $\quad f'(\pi/4) = \sqrt{2}/2$

$f''(x) = -\sin x$ $\quad f''(\pi/4) = -\sqrt{2}/2$

$f'''(x) = -\cos x$ $\quad f'''(\pi/4) = -\sqrt{2}/2$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}(\pi/4) = \sqrt{2}/2$

$$\Rightarrow f(x) = \sin x = \sqrt{2}/2 + \sqrt{2}/2\,(x-\pi/4) + \frac{-\sqrt{2}/2}{2!}(x-\tfrac{\pi}{4})^2 + \frac{-\sqrt{2}/2}{3!}(x-\pi/4)^3 + \frac{\sqrt{2}/2}{4!}(x-\tfrac{\pi}{4})^4 + \cdots$$

$$\boxed{\sin x = \frac{\sqrt{2}}{2}\left[1 + (x-\tfrac{\pi}{4}) - \frac{1}{2!}(x-\tfrac{\pi}{4})^2 - \frac{1}{3!}(x-\tfrac{\pi}{4})^3 + \frac{1}{4!}(x-\tfrac{\pi}{4})^4 + \cdots\right]}$$

we already know $f(x)=\sin x$ Taylor series converges for all $x \in \mathbb{R}$; it's still true here, even w/ different center value.

EX 5 Use what we already know to write a Maclaurin series (5 terms)

for $f(x) = \dfrac{1}{1-\sin x}$.

Remember: $\dfrac{1}{1-w} = \sum_{n=0}^{\infty} w^n \qquad |w|<1$

$$\frac{1}{1-\sin x} = \sum_{n=0}^{\infty} (\sin x)^n$$

want up to x^4 terms

$$= 1 + \sin x + \sin^2 x + \sin^3 x + \sin^4 x + \dots$$

$$= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)^2 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^3 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)^4 + \dots$$

$$= 1 + x - \frac{x^3}{6} + \left(x - \frac{x^3}{6} + \dots\right)^2 + \left(x - \frac{x^3}{6} + \dots\right)^3 + \left(x - \frac{x^3}{6} + \dots\right)^4 + \dots$$

$$= 1 + x - \frac{x^3}{6} + x^2 - \frac{2x^4}{6} + \frac{x^6}{36} + \left(x^2 - \frac{x^4}{3} + \dots\right)\left(x - \frac{x^3}{6} + \dots\right) + \left(x^4 + \dots\right) + \dots$$

$$= 1 + x - \underbrace{\frac{x^3}{6}} + x^2 - \underline{\underline{\frac{x^4}{3}}} + \underbrace{x^3} + \dots + \underline{\underline{x^4}} + \dots$$

$$\boxed{\frac{1}{1-\sin x} = 1 + x + x^2 + \frac{5}{6}x^3 + \frac{2}{3}x^4 + \dots \qquad \text{converges when } |\sin x| < 1}$$

Ex 1 (finish) Prove

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \text{ for all } x.$$

Pf $R_n(x) = \dfrac{f^{(n+1)}(c)\; x^{n+1}}{(n+1)!}$

$c \in (x,0)$ (or $c \in (0,x)$)

$\Rightarrow |R_n(x)| = \dfrac{|f^{(n+1)}(c)\; x^{n+1}|}{(n+1)!}$

$f^{(n+1)}(c) = \pm \sin c$ or $\pm \cos c$

$\Rightarrow |f^{(n+1)}(c)| = |\sin c| \le 1$ or $|\cos c| \le 1$

$0 \le |R_n(x)| \le \dfrac{|x|^{n+1}}{(n+1)!}$

[hope: $\lim_{n\to\infty} \dfrac{|x|^{n+1}}{(n+1)!} = 0$]

we know if $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges, then $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

try ART: $\lim_{n\to\infty} \dfrac{|x|^{n+1}}{(n+1)!} \cdot \dfrac{n!}{|x|^n}$

$$= |x| \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cancel{n!}} = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

for all x

$\Rightarrow$ this series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ is absolutely convergent for all $x \in (-\infty, \infty)$.

$\Rightarrow \lim_{n\to\infty} \frac{x^n}{n!} = 0 \Rightarrow \lim_{n\to\infty} |R_n(x)| = 0$

$\Rightarrow \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ converges for all x.

Conclusion

To create Taylor Series:

for $f(x)$ centered at $x=a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

at $(a-r, a+r)$