If

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

or

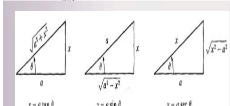
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

Then

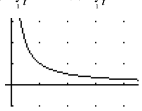
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &\quad + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \frac{f''''(x_0)}{4!}(x - x_0)^4 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \end{aligned}$$



$$\ln(x) = \int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2) = \int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv - \int v\,du$$

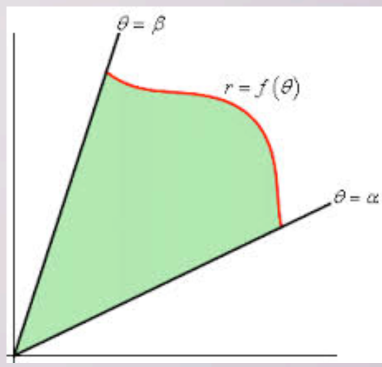
where it comes from: | the product rule for differentiation | $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

put into reverse | $\int \frac{d}{dx}(uv) = \int (u\frac{dv}{dx} + v\frac{du}{dx})$

$uv = \int u\frac{dv}{dx} + \int v\frac{du}{dx}$

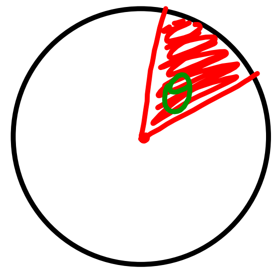
and then rearranged | $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

Calculus in Polar Coordinates



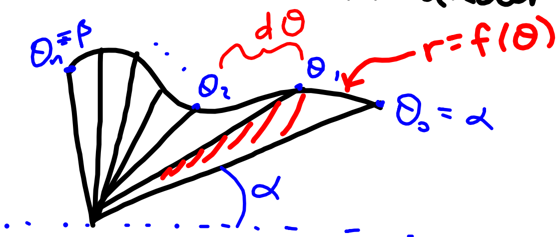
Calculus in Polar Coordinates

Begin with the area of a sector of a circle:

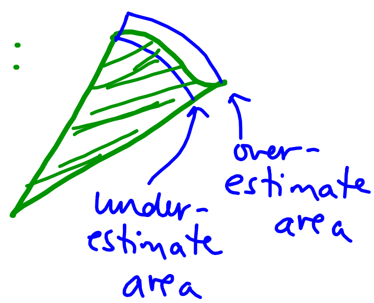


$$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{1}{2}\theta r^2$$

To find area under a curve in the plane



zoom in:



$$\Rightarrow A \approx \sum_{i=1}^{n} \frac{1}{2} r_i^2 \, d\theta$$

take limit: $A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} r_i^2 \, d\theta$

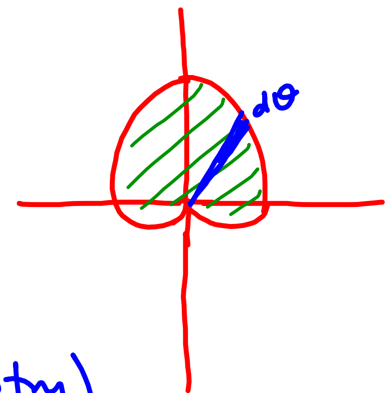
$$A = \frac{1}{2}\int_\alpha^\beta r^2 \, d\theta$$

$$\boxed{A = \frac{1}{2}\int_\alpha^\beta (f(\theta))^2 \, d\theta}$$

where $r = f(\theta)$ is the curve

EX 1 Find the area inside $r = 3 + 3\sin\theta$

Cardioid



$$A = \frac{1}{2}\int_{\alpha}^{\beta} (3+3\sin\theta)^2\, d\theta$$

$$0 \le \theta \le 2\pi$$

(take advantage of symmetry)

$$A = \frac{1}{2}(2)\int_{-\pi/2}^{\pi/2} (3+3\sin\theta)^2\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} (9 + 18\sin\theta + 9\sin^2\theta)\, d\theta$$

$$= (9\theta - 18\cos\theta)\Big|_{-\pi/2}^{\pi/2} + \frac{9}{2}\int_{-\pi/2}^{\pi/2} (1 - \cos(2\theta))\, d\theta$$

$$= \left(9\left(\pi/2 - -\pi/2\right) - 18\left(\cancel{\cos(\pi/2) - \cos(-\pi/2)}^{\,0}\right)\right)$$

$$+ \frac{9}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]_{-\pi/2}^{\pi/2}$$

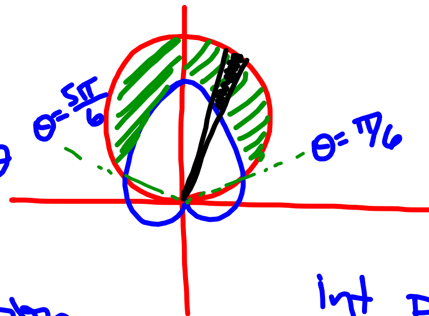
$$= 9\pi + \frac{9}{2}\left(\pi/2 - -\pi/2\right) - \frac{9}{4}\left(\cancel{\sin(\pi) - \sin(-\pi)}^{\,0}\right)$$

$$= 9\pi + \frac{9}{2}(\pi) = \frac{27}{2}\pi$$

EX 2 Find the area inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$.
circle cardioid

$$A = \frac{1}{2}\int_\alpha^\beta (f(\theta))^2\, d\theta$$

$$= \frac{1}{2}\int_\alpha^\beta \left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] d\theta$$



to get limits of integration, we need intersectn pts!!!

int pts

$3\sin\theta = 1+\sin\theta$

$2\sin\theta = 1$

$\sin\theta = 1/2$

$\theta = \pi/6,\ 5\pi/6$

$$A = \frac{1}{2}(2)\int_{\pi/6}^{\pi/2}\left[9\sin^2\theta - (1+2\sin\theta + \sin^2\theta)\right] d\theta$$

$$= \int_{\pi/6}^{\pi/2}\left[8\sin^2\theta - 1 - 2\sin\theta\right] d\theta$$

$$= \int_{\pi/6}^{\pi/2}\left[\frac{8}{2}(1-\cos(2\theta)) - 1 - 2\sin\theta\right] d\theta$$

$$= \int_{\pi/6}^{\pi/2}(3 - 4\cos(2\theta) - 2\sin\theta)\, d\theta$$

$$= \left(3\theta - \frac{4}{2}\sin(2\theta) + 2\cos\theta\right)\Big|_{\pi/6}^{\pi/2}$$

$$= \left(\frac{3\pi}{2} - 2\underset{0}{\sin\pi} + 2\underset{0}{\cos(\pi/2)}\right) - \left(3\left(\frac{\pi}{6}\right) - 2\sin(\pi/3) + 2\cos(\pi/6)\right)$$

$$= \frac{3\pi}{2} - \frac{\pi}{2} + 2\left(\cancel{\frac{\sqrt{3}}{2}}\right) - 2\left(\cancel{\frac{\sqrt{3}}{2}}\right)$$

$$= \pi$$

Tangent line slope on a polar curve

$m = \frac{dy}{dx}$ in rectangular coordinates

polar coords $\Rightarrow \begin{cases} y = r\sin\theta = f(\theta)\sin\theta \\ x = r\cos\theta = f(\theta)\cos\theta \end{cases}$

$r = f(\theta)$

$$\Rightarrow \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

and $\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

slope of a polar curve.

EX 3 Find the slope of the tangent line to $r = \underbrace{2\text{-}3\sin\theta}_{f(\theta)}$ at $\theta = \pi/6$.

$$m = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$f'(\theta) = -3\cos\theta$

$$= \frac{(2-3\sin\theta)\cos\theta + -3\cos\theta\sin\theta}{-3\cos^2\theta - (2-3\sin\theta)\sin\theta}$$

Slope at $\theta = \pi/6$ is $m\Big|_{\pi/6} = \dfrac{\left(2-3\left(\frac{1}{2}\right)\right)\frac{\sqrt{3}}{2} - 3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{-3\left(\frac{\sqrt{3}}{2}\right)^2 - \left(2-3\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)}$

$= \sqrt{3}/5$

EX 4 For what values of θ will the tangent line to $r = 2\text{-}3\sin\theta$ be horizontal?

$\theta = ?$ when $m = 0$

$$0 = \frac{(2-3\sin\theta)\cos\theta - 3\cos\theta\sin\theta}{-3\cos^2\theta - (2-3\sin\theta)\sin\theta}$$

$$0 = \frac{2\cos\theta - 6\sin\theta\cos\theta}{-3\cos^2\theta + 3\sin^2\theta - 2\sin\theta}$$

$0 = 2\cos\theta - 6\sin\theta\cos\theta$

$0 = 2\cos\theta(1 - 3\sin\theta)$

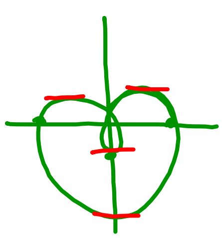
$2\cos\theta = 0$ or $1 - 3\sin\theta = 0$

$\theta = \pi/2, 3\pi/2$

$\sin\theta = 1/3$

$\theta = \arcsin(1/3)$, $\pi - \arcsin(1/3)$

$r = 2-3\sin\theta$



Conclusion:

- Intro to some calculus topics in polar coords (area "under" curve, and slope)