If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

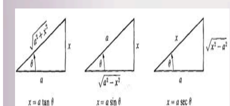
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

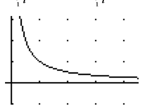
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

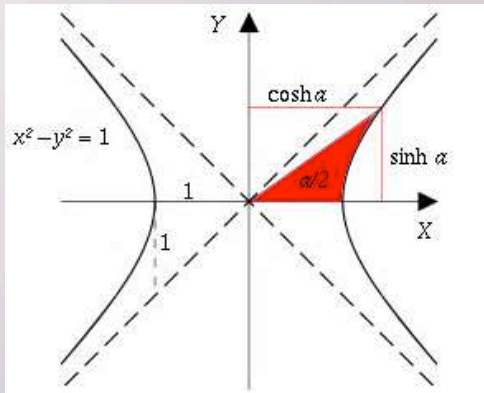
$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

The Hyperbolic Functions



Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

(called "sinsh")

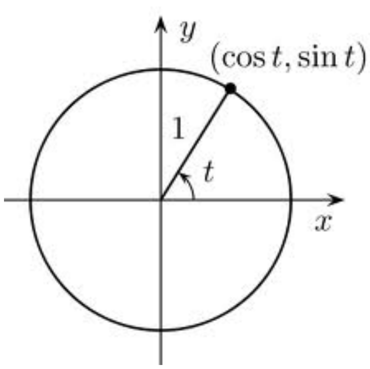
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Relation to the trigonometric functions:

Trigonometry Fns

1. (cos θ, sin θ) is a point on the unit circle.
2. y= sin x is an odd function.
3. y = cos x is an even function.
4. $\sin^2\theta+\cos^2\theta=1$



odd fn

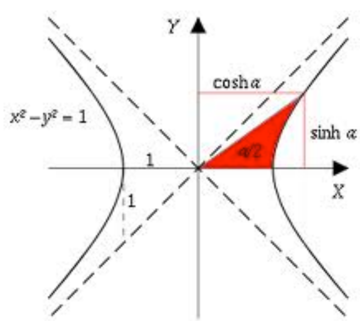
$f(-x)=-f(x)$

even fn

$f(-x)=f(x)$

Hyperbolic Fns

1. (cosh θ, sinh θ) is a point on the unit hyperbola.
2. y = sinh x is an odd function.
3. y = cosh x is an even function.
4. $\cosh^2\theta - \sinh^2\theta=1$



Prove $\cosh^2\theta - \sinh^2\theta = 1$.

note: $\sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2}$ defns

$\cosh\theta = \dfrac{e^{\theta} + e^{-\theta}}{2}$

Pf

$\cosh^2\theta - \sinh^2\theta$

$$= \left(\frac{e^{\theta}+e^{-\theta}}{2}\right)^2 - \left(\frac{e^{\theta}-e^{-\theta}}{2}\right)^2$$

$$= \frac{1}{4}\left[\left(e^{\theta}+e^{-\theta}\right)^2 - \left(e^{\theta}-e^{-\theta}\right)^2\right]$$

$$= \frac{1}{4}\left[e^{2\theta} + \underbrace{e^{\theta}e^{-\theta}}_{1} + \underbrace{e^{\theta}e^{-\theta}}_{1} + e^{-2\theta} - \left(e^{2\theta} - \underbrace{e^{\theta}e^{-\theta}}_{1} - \underbrace{e^{\theta}e^{-\theta}}_{1} + e^{-2\theta}\right)\right]$$

$$= \frac{1}{4}\left[\cancel{e^{2\theta}} + 2 + \cancel{e^{-2\theta}} \cancel{- e^{2\theta}} + 2 \cancel{- e^{-2\theta}}\right]$$

$$= \frac{1}{4}[4] = 1$$

#

$D_x(\sinh x) = \cosh x$ $\qquad$ $D_x(\cosh x) = \sinh x$

$D_x(\tanh x) = \text{sech}^2 x$ $\qquad$ $D_x(\coth x) = -\text{csch}^2 x$

$D_x(\text{sech } x) = -\text{sech } x \tanh x$ $\qquad$ $D_x(\text{csch } x) = -\text{csch } x \coth x$

EX 1 $D_x(\underbrace{\coth(4x)}_{1}\underbrace{\sinh x}_{2}) = -\text{csch}^2(4x)(4)\sinh x$

$+ \coth(4x)\cosh x$

EX 2 $\int \tanh x \ln(\cosh x)\,dx = \int u\,du$

$u = \ln(\cosh x)$

$du = \frac{1}{\cosh x}(\sinh x)\,dx$

$du = \tanh x\,dx$

$= \frac{u^2}{2} + C$

$= \frac{1}{2}(\ln(\cosh x))^2 + C$

<u>note</u>: $(\ln(\cosh x))^2$

$\neq \ln(\cosh x)^2$

$= 2\ln(\cosh x)$

EX 3 Verify this identity.

$$\tanh(x-y)=\frac{\tanh x-\tanh y}{1-\tanh x\tanh y} \qquad \text{Hint:} \quad \tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

Pf

$$\frac{\tanh x-\tanh y}{1-\tanh x\tanh y}=\frac{\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)-\left(\frac{e^y-e^{-y}}{e^y+e^{-y}}\right)}{1-\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)\left(\frac{e^y-e^{-y}}{e^y+e^{-y}}\right)}\cdot\frac{(e^x+e^{-x})(e^y+e^{-y})}{(e^x+e^{-x})(e^y+e^{-y})}$$

$$=\frac{(e^x-e^{-x})(e^y+e^{-y})-(e^y-e^{-y})(e^x+e^{-x})}{(e^x+e^{-x})(e^y+e^{-y})-(e^x-e^{-x})(e^y-e^{-y})}$$

$$=\frac{\cancel{e^{x+y}}+e^{x-y}-e^{-x+y}\cancel{-e^{-x-y}}\cancel{-e^{x+y}}-e^{-x+y}+e^{-y+x}\cancel{+e^{-y-x}}}{\cancel{e^{x+y}}+e^{x-y}+e^{-x+y}\cancel{+e^{-x-y}}\cancel{-e^{x+y}}+e^{x-y}+e^{-x+y}\cancel{-e^{-x-y}}}$$

$$=\frac{2e^{x-y}-2e^{y-x}}{2e^{x-y}+2e^{y-x}}$$

$$=\frac{\cancel{2}(e^{x-y}-e^{y-x})}{\cancel{2}(e^{x-y}+e^{y-x})}=\frac{e^{x-y}-e^{-(x-y)}}{e^{x-y}+e^{-(x-y)}}$$

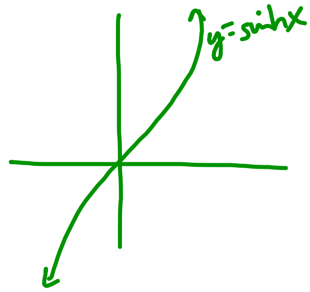
$$=\tanh(x-y) \ \#$$

note
$y-x$
$=-(x-y)$

Inverse Hyperbolic Functions

Let $y = \sinh x \Leftrightarrow x = \sinh^{-1} y$ (if the inverse exists).

domain/range $\mathbb{R}$



Find $x = \sinh^{-1} y$.

$$y = \sinh x = \frac{e^x - e^{-x}}{2}$$

$$2y = e^x - e^{-x}$$

goal: get x by itself. (solve for x)

$$e^x(2y) = \left(e^x - \frac{1}{e^x}\right)e^x$$

$$2e^x y = e^{2x} - 1$$

$$0 = e^{2x} - 2ye^x - 1$$

use quadratic formula

$$0 = (e^x)^2 - 2y(e^x) - 1$$

$$e^x = \frac{2y \pm \sqrt{(-2y)^2 - 4(1)(-1)}}{2(1)}$$

$$e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$$

$$= \frac{2y \pm \sqrt{4(y^2+1)}}{2}$$

$$e^x = \frac{2y \pm 2\sqrt{y^2+1}}{2}$$

$$e^x = y \pm \sqrt{y^2+1}$$

know $e^x > 0$ for all x-values

and $\sqrt{y^2+1} > \sqrt{y^2} = |y|$

$\Rightarrow y - \sqrt{y^2+1} < 0$

throw this "solution" away

$$\Rightarrow e^x = y + \sqrt{y^2+1}$$

$$x = \ln\left(y + \sqrt{y^2+1}\right)$$

$$\Rightarrow \boxed{\sinh^{-1}(y) = \ln\left(y + \sqrt{y^2+1}\right)}$$

domain/range $\mathbb{R}$

$$\sinh^{-1} x = \ln(x+\sqrt{x^2+1}) \qquad \cosh^{-1} x = \ln(x+\sqrt{x^2-1}) \quad x \ge 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad x \in (-1,1) \qquad \operatorname{sec}h^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \quad x \in (0,1]$$

How do we find the derivative of these functions?

$$D_x(\sinh^{-1} x) = D_x\left(\ln\left(x+\sqrt{x^2+1}\right)\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{\cancel{2}}(x^2+1)^{-1/2}(\cancel{2}x)\right)$$

$$= \frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}\right)$$

$$= \frac{\cancel{(\sqrt{x^2+1}+x)}}{\cancel{(x+\sqrt{x^2+1})}(\sqrt{x^2+1})}$$

$$= \frac{1}{\sqrt{x^2+1}}$$

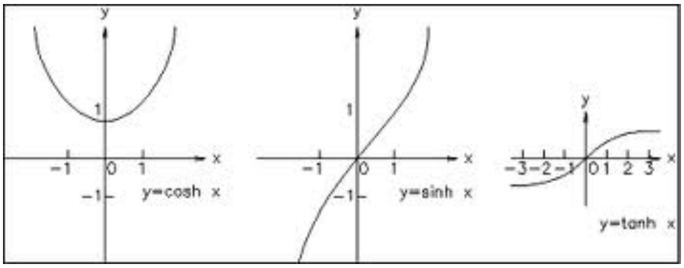
$$D_x\left(\sinh^{-1} x\right)=\frac{1}{\sqrt{x^2+1}} \qquad D_x\left(\cosh^{-1} x\right)=\frac{1}{\sqrt{x^2-1}} \quad x>1$$

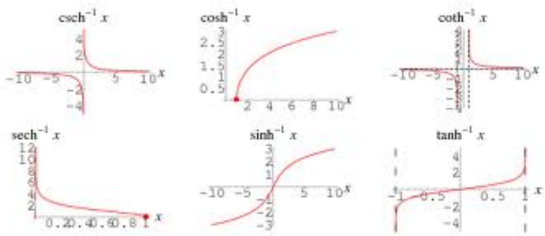
$$D_x\left(\tanh^{-1} x\right)=\frac{1}{1-x^2} \quad x\in(-1,1) \qquad D_x\left(\operatorname{sech}^{-1} x\right)=\frac{-1}{x\sqrt{1-x^2}} \quad x\in(0,1)$$

EX 4 Find y' .

$$y = \overset{1}{x^2}\, \overset{2}{\sinh^{-1}(x^5)}$$

$$y' = 2x\sinh^{-1}(x^5) + x^2\left(\frac{1}{\sqrt{(x^5)^2+1}}\right)(5x^4)$$





Summary

- defn of hyperbolic fns
- inverse hyperbolic fns
- derivative/integral formulas for hyperbolic fns.