If

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

or

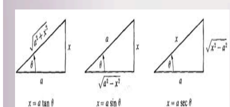
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

Then

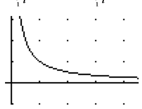
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \frac{f''''(x_0)}{4!}(x - x_0)^4 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$



$$\ln(x) = \int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2) = \int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv - \int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

put into reverse

$$\int \frac{d}{dx}(uv) = \int \left(u\frac{dv}{dx} + v\frac{du}{dx}\right)$$

$$uv = \int u\frac{dv}{dx} + \int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$$

Basic Integration Rules

$f(x)$	$\int f(x)\ dx$
x^n	$\frac{x^{n+1}}{n+1} + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
$\frac{f'(x)}{f(x)}$	$\ln\lvert f(x)\rvert + c$

Basic Integration Rules: Substitution

u-substitution for Integration

Let g be a differentiable function and suppose F is an antiderivative of f.

If $u = g(x)$, then $\int f(g(x))g'(x)dx = \int f(u)du = F(u) + c = F(g(x))+c$.

EX 1 $\int \frac{3x}{\sin^2(4x^2)}dx = 3\int x \csc^2(4x^2)\,dx$

$u = 4x^2$

$du = 8x\,dx$

$\frac{1}{8}du = x\,dx$

$= 3\left(\frac{1}{8}\right)\int \csc^2(u)\,du$

$= \frac{3}{8}(-\cot u) + C$

$= \boxed{\frac{-3}{8}\cot(4x^2) + C}$

EX 2 $\int \frac{5e^{3/x^2}}{x^3}dx$

$u = \frac{3}{x^2} = 3x^{-2}$

$du = -6x^{-3}\,dx$

$-\frac{1}{6}du = \frac{1}{x^3}dx$

$= -\frac{1}{6}\int 5e^u\,du$

$= \frac{-5}{6}\int e^u\,du = \frac{-5}{6}(e^u) + C$

$= \boxed{\frac{-5}{6}\left(e^{3/x^2}\right) + C}$

EX 3 $\int \frac{5}{9+(2x-1)^2}\,dx$ $= \frac{1}{2}\int \frac{5}{9+u^2}\,du$

$\int \frac{1}{1+p^2}\,dp = \arctan(p) + C$

$u = 2x-1$
$du = 2\,dx$
$\frac{1}{2}du = dx$

$= \frac{5}{2}\int \frac{1}{9+u^2}\,du$

$= \frac{5}{2}\int \frac{1}{9}\left(\frac{1}{1+\left(\frac{u}{3}\right)^2}\right)du$

$w = \frac{u}{3}$
$dw = \frac{1}{3}du$
$3dw = du$

$= \frac{5}{18}\int \frac{1}{1+\left(\frac{u}{3}\right)^2}\,du$

$= \frac{5}{18}\int \frac{1}{1+w^2}(3)\,dw$ $= \frac{5}{6}\int \frac{1}{1+w^2}\,dw$

$= \frac{5}{6}\arctan(w) + C$

$= \boxed{\frac{5}{6}\arctan\left(\frac{2x-1}{3}\right) + C}$

note (Ex 3)
we could have chosen $u = \frac{2x-1}{3}$ at start.

EX 4 $\int \frac{3x^2-4x+2}{x-2}\,dx$

(this is improper rational fn)
do long division

$= \int \left(3x+2+\frac{6}{x-2}\right)dx$

$$\begin{array}{r} 3x+2 \\ x-2\,\overline{)\,3x^2-4x+2} \\ -(3x^2-6x) \\ \hline 2x+2 \\ -(2x-4) \\ \hline 6 \end{array}$$

$= \frac{3x^2}{2} + 2x + 6\int \frac{1}{x-2}\,dx$

$= \frac{3x^2}{2} + 2x + 6\int \frac{1}{u}\,du$

$= \frac{3x^2}{2} + 2x + 6\ln|u| + C$

$= \boxed{\frac{3x^2}{2} + 2x + 6\ln|x-2| + C}$

Fact:
$\int \frac{1}{x}\,dx = \ln|x| + C$
$\int \frac{1}{\heartsuit}\,d\heartsuit = \ln|\heartsuit| + C$

$u = x-2$
$du = dx$

Note: (shortcut)

$\int \frac{1}{mx+b}\,dx$ (m, b constants, $m \neq 0$)

$u = mx+b$
$du = m\,dx$
$\frac{1}{m}du = dx$

$= \frac{1}{m}\int \frac{1}{u}\,du$

$= \frac{1}{m}\ln|u| + C = \frac{1}{m}\ln|mx+b| + C$

shortcut: $\boxed{\int \frac{1}{mx+b}\,dx = \frac{1}{m}\ln|mx+b| + C}$

"The integral of 1 over a linear polynomial is ln of abs. value of that linear polynomial divided by leading coefficient."

EX 5 $\int \frac{2x}{\sqrt{1-x^4}}dx$

$u = 1 - x^4$

$du = -4x^3\,dx$

idea doesn't work

$\frac{-1}{4}du = x^3\,dx$

(inside integral, I have $x\,dx$, but not $x^3\,dx$)

$\frac{-1}{4}\left(\frac{1}{x^2}\right)du = x\,dx$

$= \int \frac{du}{\sqrt{1-u^2}}$

$u = x^2 \Rightarrow x^4 = (x^2)^2 = u^2$

$du = 2x\,dx$

$= \int \frac{1}{\sqrt{1-u^2}}du$

$= \arcsin u + C$

$= \boxed{\arcsin(x^2) + C}$

EX 6 $\int \frac{\sin(\ln(4x^2))}{x} dx$

(note: I don't know how to integrate ln fn, but I do know how to differentiate it!)

let $u = \ln(4x^2)$

$du = \frac{1}{4x^2}(8x)\,dx$

$du = \frac{2}{x}\,dx$

$\frac{1}{2}\,du = \frac{1}{x}\,dx$

$= \frac{1}{2}\int \sin(u)\,du = \frac{1}{2}(-\cos u) + C$

$= \boxed{\frac{-1}{2}\cos\left(\ln(4x^2)\right) + C}$

In conclusion

u-substitution for integration
(difficult, creative process
that requires a lot of practice)