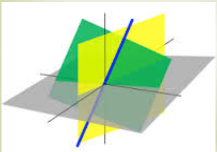
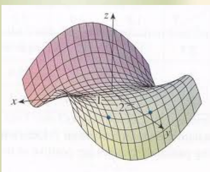


$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

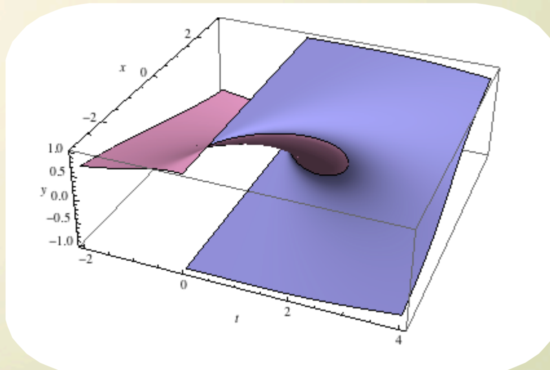


$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

Limits and Continuity



Review from Calculus 1

Definition: Continuity at a Point

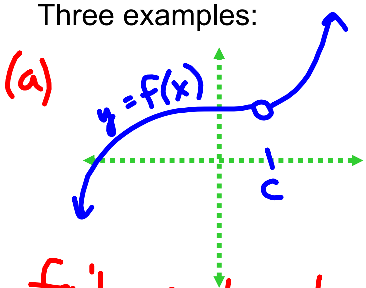
Let f be defined on an open interval containing c. We say that f is continuous at c if

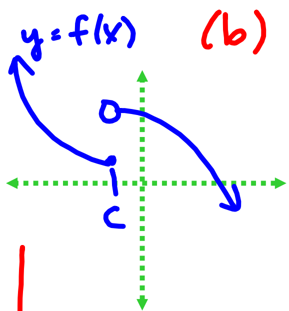
$$\lim_{x\to c} f\left(x\right) = f\left(c\right)$$

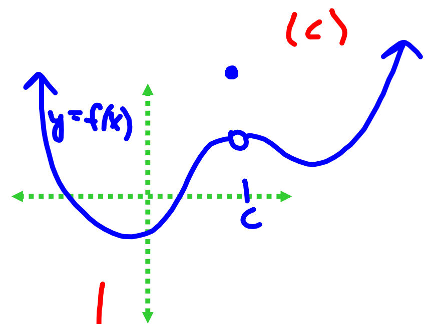
This indicates three things:

1. The function is defined at $x = c$.
2. The limit exists at $x = c$.
3. The limit at $x = c$ needs to be exactly the value of the function at x = c.

Three examples:







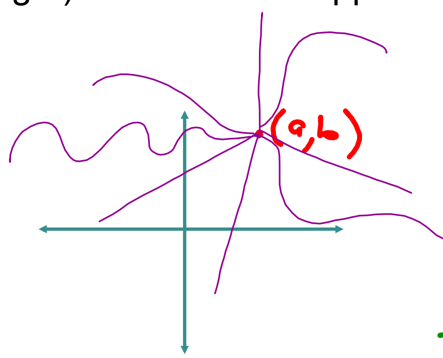
fails continuity at $x = c$ because $f(c)$ DNE

discontinuous at $x = c$ because limit DNE at $x = c$

discontinuous at $x = c$ because $f(c) \neq \lim_{x\to c} f(x)$

Limits and Continuity

Intuitively, $\lim_{(x,y)\to(a,b)} f(x,y) = L$ means that as the point (x,y) gets very close to (a,b), then $f(x,y)$ gets very close to L. When we did this for functions of one variable, it could approach from only two sides or directions (left or right). Now we can approach (a,b) from infinitely many directions.



there are infinitely many paths to (a,b) in xy plane

Definition of Limit

$\lim_{(x,y)\to(a,b)} f(x,y) = L$ means that for all $\varepsilon > 0$ there exists a corresponding $\delta > 0$ such that $|f(x,y) - L| < \varepsilon$ provided that $0 < |(x,y)-(a,b)| < \delta$. We can make $f(x,y)$ as close as we'd like to L by choosing (x,y) sufficiently close to (a,b).

note:

$$|(x,y)-(a,b)| = \sqrt{(x-a)^2 + (y-b)^2}$$

EX 1 Find $\lim_{(x,y)\to(-2,1)} \left(xy^3 - xy + 3y^2\right)$.

$= (-2(1^3) - (-2)(1) + 3(1^2))$

$= -2 + 2 + 3$

$= 3$

EX 2 Find $\lim_{(x,y)\to(0,0)} \frac{1+xy}{\cos(xy)}$.

$= \frac{1+0(0)}{\cos(0)} = \frac{1}{1} = 1$

Strategies for finding a limit

(1) try plugging in the numbers

(2) manipulate algebraically into something we recognize (from previous knowledge)

(3) try approaching the pt from 2 different paths

(4) use polar coordinates

EX 3 $\lim_{(x,y)\to(0,0)} \dfrac{\tan(x^2+y^2)}{x^2+y^2}$ $= \lim_{(x,y)\to(0,0)} \left(\dfrac{\sin(x^2+y^2)}{(x^2+y^2)}\right)\left(\dfrac{1}{\cos(x^2+y^2)}\right)$

try plugging in $x=0$, and $y=0$

$\dfrac{\tan 0}{0} = \dfrac{0}{0}$ case

$\Rightarrow f(x,y) = \dfrac{\tan(x^2+y^2)}{x^2+y^2}$ is undefined at $(0,0)$

note: $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

$= \lim_{x^2+y^2\to 0} \dfrac{\sin(x^2+y^2)}{(x^2+y^2)} = 1$

$= 1\left(\frac{1}{1}\right) = 1$

note: $\lim_{♡\to 0} \dfrac{\sin ♡}{♡} = 1$

EX 4 Show that $\lim_{(x,y)\to(0,0)} \dfrac{xy+y^3}{x^2+y^2}$ does not exist.

try 2 different paths:

① $x=y$ (a line that goes through $(0,0)$)

$\lim_{y\to 0} \dfrac{y^2+y^3}{y^2+y^2}$

$= \lim_{y\to 0} \dfrac{\cancel{y^2}(1+y)}{2\cancel{y^2}}$

$= \lim_{y\to 0} \dfrac{1+y}{2} = \dfrac{1}{2}$

② $y=x^2$ (a curve that goes thru $(0,0)$)

$\lim_{x\to 0} \dfrac{x(x^2)+(x^2)^3}{x^2+(x^2)^2} =$

$\lim_{x\to 0} \dfrac{x^3+x^6}{x^2+x^4}$

$= \lim_{x\to 0} \dfrac{x^3(1+x^3)}{x^2(1+x^2)}$

$= \lim_{x\to 0} \dfrac{x(1+x^3)}{1+x^2} = \dfrac{0(1)}{1} = 0$

$\Rightarrow$ limit DNE

EX 5 Find the limits.

a) $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$ *Hint: Use polar coordinates.*

$r^2 = x^2 + y^2$

$x = r\cos\theta$

$y = r\sin\theta$

$= \lim_{r\to 0} \frac{(r\cos\theta)^3}{r^2}$

$= \lim_{r\to 0} (r\cos^3\theta) = \cos^3\theta \left(\lim_{r\to 0} r\right)$

$= 0$

note: $(x,y)\to(0,0) \iff x^2+y^2 \to 0$

b) $\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$

$= \lim_{r\to 0} \frac{r\cos\theta}{r^2}$

$= \lim_{r\to 0} \frac{\cos\theta}{r}$ DNE

c) $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta}{r^2} = \lim_{r\to 0} \cos^2\theta$

$= \cos^2\theta$

(θ can be literally any value)

$\Rightarrow \cos^2\theta$ can be different values

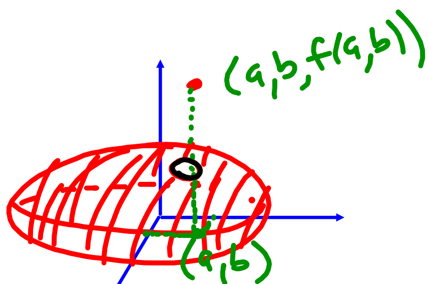
$\Rightarrow$ limit DNE

Continuity

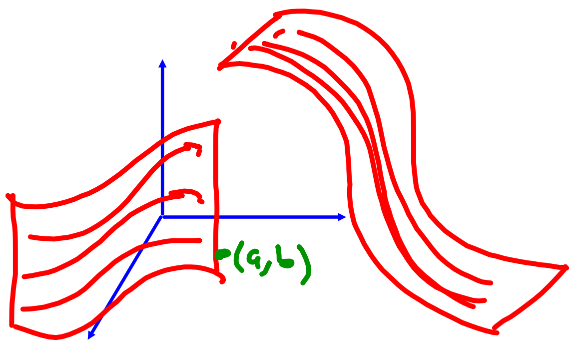
A function $f(x,y)$ is continuous at (a,b) if $\boxed{f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)}$

This indicates three things:

a) the function is defined at (a,b),
b) the limit of f as $(x,y)\to(a,b)$ exists, and
c) the limit of f at (a,b) is exactly the same as $f(a,b)$.



$\lim_{(x,y)\to(a,b)} f(x,y)$ exists and $f(a,b)$ exists but they are not the same



$\lim_{(x,y)\to(a,b)} f(x,y)$ DNE but $f(a,b)$ exists

Composition of Functions

If a function, g, of two variables is continuous at (a,b) and a function, f, of one variable is continuous at $g(a,b)$, then

$(f \circ g)(x,y) = f(g(x,y))$ is continuous at (a,b).

EX 6 Show that $f(x,y) = sin(x^2-4xy)$ is continuous everywhere.

$g(x,y) = x^2-4xy$ is polynomial in 2 vars
and continuous everywhere
and $h(w) = \sin w$ is also continuous
fn $\Rightarrow h(g(x,y))$ is continuous

EX 7 Determine where $f(x,y) = \ln(1-x^2-y^2)$ is continuous.

require that $1-x^2-y^2 > 0$
$\Leftrightarrow x^2+y^2 < 1$

EX 8 Is $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{xy} & if\ xy \neq 0 \\ 1 & if\ xy = 0 \end{cases}$ continuous everywhere?

piece 1:

$\frac{\sin(xy)}{(xy)}$ has problems if $xy=0$ ($\Leftrightarrow x=0$ or $y=0$)

need to find $\lim_{xy\to 0} \frac{\sin(xy)}{xy} = 1$

$\Rightarrow$ limit and the fn-value are both 1 as $xy \to 0$

$\Rightarrow$ fn is continuous