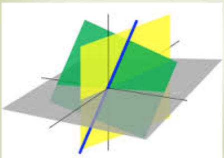
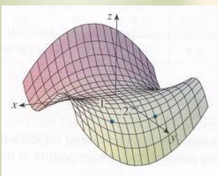
Differentiability/Gradient



$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

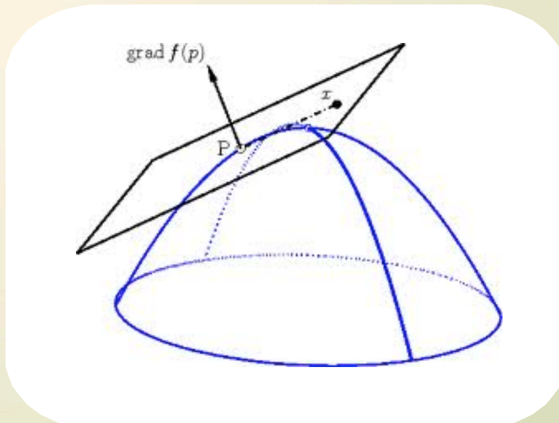
$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3 dy$$

$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$



Differentiability

For a function of one variable, the derivative gives us the slope of the tangent line, and a function of one variable is differentiable if the derivative exists. For a function of two variables, the function is differentiable at a point if it has a tangent plane at that point. But existence of the first partial derivatives is not quite enough, unlike the one-variable case.

Theorem

If $f(x,y)$ has continuous partial derivatives $f_x(x,y)$ and $f_y(x,y)$ on a disk D whose interior contains (a,b), then $f(x,y)$ is differentiable at (a,b).

Theorem

If f is differentiable at (a,b), then f is continuous at (a,b).

differentiability $\Rightarrow$ continuity

Gradient of f

$$\nabla f(p) = \nabla f(a,b) = \left\langle f_x(a,b), f_y(a,b) \right\rangle = f_x(a,b)\hat{i} + f_y(a,b)\hat{j}$$

for a function, $z = f(x,y)$.
(Note: This gradient lives in 2-D space, but it is the gradient of a function whose graph is 3-D.)

gradient is a vector!!

Properties of Gradient Operator
p is the input point (a,b).

$$\left.\begin{aligned} &\nabla[f(p)+g(p)] = \nabla f(p) + \nabla g(p) \\ &\nabla[\alpha f(p)] = \alpha \nabla f(p), \alpha \in \Re \end{aligned}\right\}$$

gradient is a linear operator

$$\nabla[f(p)g(p)] = f(p)\nabla g(p) + \nabla f(p) g(p)$$

("product rule")

EX 1 Find the gradient of *f*.

a) $f(x,y) = x^3y - y^3$

$$\nabla f = f_x \hat{\imath} + f_y \hat{\jmath} = (3x^2y)\hat{\imath} + (x^3 - 3y^2)\hat{\jmath}$$

b) $f(x,y) = \sin^3(x^2y)$

$$\nabla f = 3\sin^2(x^2y)(\cos(x^2y))(2xy)\hat{\imath}$$
$$+ 3\sin^2(x^2y)(\cos(x^2y))(x^2)\hat{\jmath}$$

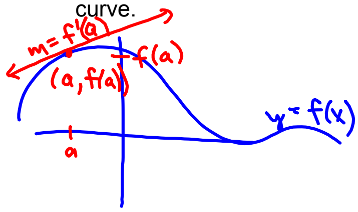
c) $f(x,y,z) = xz\ln(x+y+z)$

$$\nabla f = f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}$$
$$= \left(z\ln(x+y+z) + \frac{xz(1)}{x+y+z}\right)\hat{\imath}$$
$$+ \left(\frac{xz(1)}{x+y+z}\right)\hat{\jmath} + \left(x\ln(x+y+z) + \frac{xz(1)}{x+y+z}\right)\hat{k}$$

Tangent Plane

Curves in 2-D

Remember the equation of the tangent line to a 2-D curve.

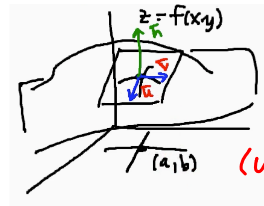


$y - f(a) = f'(a)(x-a)$

$$\boxed{y = f(a) + f'(a)(x-a)}$$

eqn of tangent line to curve $y=f(x)$ (in 2-d)

Surfaces in 3-D



(input pt)

find $\vec{u}$ and $\vec{v}$ (vectors in the tangent plane)

$\Rightarrow \vec{n} = \vec{u} \times \vec{v}$

$\vec{u}$ = no "y-movement"

$= \langle 1, 0, f_x(a,b) \rangle$

$\vec{v}$ = no "x-movement"

$= \langle 0, 1, f_y(a,b) \rangle$

$$\vec{n} = \vec{u} \times \vec{v}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$$

$$= \hat{\imath}\,(-f_x(a,b)) - \hat{\jmath}\,(f_y(a,b)) + \hat{k}\,(1)$$

$$\vec{n} = \langle -f_x(a,b), -f_y(a,b), 1 \rangle$$

$\Rightarrow$ eqn of plane w/ $\vec{n}$ normal vector and through pt $(a,b,f(a,b))$

$$\langle -f_x(a,b), -f_y(a,b), 1 \rangle \cdot \langle x-a, y-b, z-f(a,b) \rangle = 0$$

$$\star \quad -f_x(a,b)(x-a) - f_y(a,b)(y-b) + z - f(a,b) = 0$$

$$\boxed{\begin{aligned} & z = f(a,b) + \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle x-a, y-b \rangle \\ \text{or} \quad & z = f(a,b) + \nabla f(a,b) \cdot \langle x-a, y-b \rangle \end{aligned}}$$

eqn of tangent plane to surface $z = f(x,y)$ at (a,b) input pt (in 3-d)

EX 2 For $f(x,y) = x^3y+3xy^2$, find the equation of the tangent plane at $(a,b) = (2,-2)$.

$$\langle f_x(a,b), f_y(a,b)\rangle \cdot \langle x-a, y-b\rangle = z - f(a,b)$$

$$\nabla f(a,b) \cdot \langle x-a, y-b\rangle = z - f(a,b)$$

$$z = f(a,b) + \nabla f(a,b) \cdot \langle x-a, y-b\rangle$$ tangent plane

$f_x = 3x^2y + 3y^2$, $f_y = x^3 + 6xy$

$f(a,b) = f(2,-2) = 8(-2) + 3(2)(4) = 8$

tangent plane: $z = 8 + \langle 3(2^2)(-2)+3(2)^2, 2^3+6(2)(-2)\rangle \cdot \langle x-2, y+2\rangle$

$z = 8 + \langle -12, -16\rangle \cdot \langle x-2, y+2\rangle$

$z = 8 + -12(x-2) - 16(y+2)$

$z = -12x - 16y$

$$\boxed{12x + 16y + z = 0}$$

Ex 3 Find the equation of the tangent "hyperplane" to $f(x,y,z) = w$ at the point (a,b,c).

$f(x,y,z) = xyz+x^2 \quad (a,b,c) = (2,0,-3)$

$$w = f(a,b,c) + \nabla f(a,b,c) \cdot \langle x-a, y-b, z-c \rangle$$

$$f(a,b,c) = f(2,0,-3) = 4$$

$$f_x = yz+2x, \quad f_y = xz, \quad f_z = xy$$

$$f_x(2,0,-3) = 4, \quad f_y(2,0,-3) = -6, \quad f_z(2,0,-3) = 0$$

$$\Rightarrow \nabla f(2,0,-3) = \langle 4,-6,0 \rangle$$

tangent hyperplane:

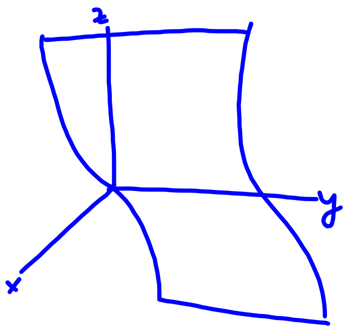
$$w = 4 + \langle 4,-6,0 \rangle \cdot \langle x-2, y, z+3 \rangle$$

$$w = 4 + 4(x-2) - 6(y) + 0(z+3)$$

$$w = 4x - 6y - 4$$

$$\boxed{4x - 6y - w = 4}$$

Ex 4 Find all domain points (x,y) at which the tangent plane to the graph of $z = x^3$ is horizontal. $z=f(x,y)=x^3$



tangent plane horizontal

$\Longleftrightarrow$ normal of tangent plane $c\langle 0,0,1\rangle$

find the tangent plane to $z=f(x,y)$ at (a,b)

$z= a^3 + \nabla f(a,b)\cdot\langle x-a, y-b\rangle$ | $f_x = 3x^2$

$z= a^3 + \langle 3a^2, 0\rangle\cdot\langle x-a, y-b\rangle$ | $f_y = 0$

$z = a^3+3a^2(x-a)+0$

$z= 3a^2x-3a^3+a^3$

$3a^2x-z=2a^3$

$\Rightarrow$ normal vector is $\langle 3a^2, 0, -1\rangle$

force $\langle 3a^2, 0, -1\rangle = c\langle 0,0,1\rangle$

$\Rightarrow$ let $c=-1$, $-1=-1$ ✓ (z-component)

$3a^2=0$

$\Rightarrow a=0$

$\Rightarrow$ tangent plane is horizontal whenever $x=0$ and if $x=0$, $f(x,y)=x^3=0$ is true.

at pts of surface on y-axis, tangent plane is horizontal.