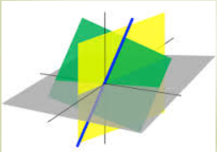
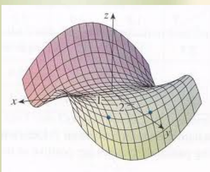


$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

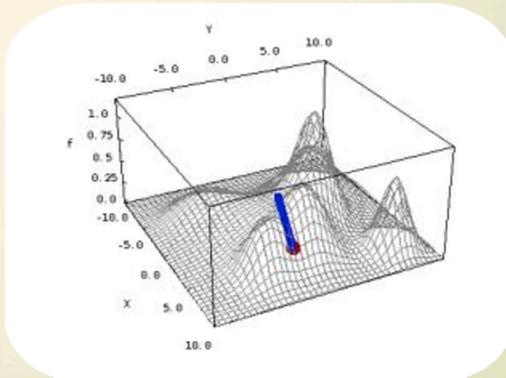


$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y \right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2} \right]_{y=0}^{y=1} = \frac{1}{2}$$

Directional Derivatives



Directional Derivatives

We know we can write
$$\frac{\partial f}{\partial x} = f_x(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$$
$$\frac{\partial f}{\partial y} = f_y(a,b) = \lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}$$

The partial derivatives measure the rate of change of the function at a point in the direction of the x-axis or y-axis. What about the rates of change in the other directions?

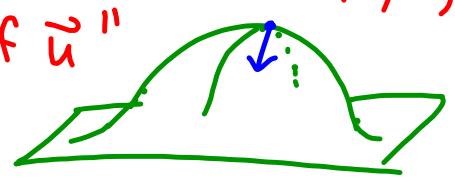
Definition

For any unit vector, $\hat{u} = \langle u_x, u_y \rangle$ let

$$D_{\vec{u}} f(a,b) = \lim_{h\to 0}\frac{f(a+hu_x, b+hu_y)-f(a,b)}{h}$$

If this limit exists, this is called the directional derivative of f at the point (a,b) in the direction of $\hat{u}$.

read "directional derivative of f at (a,b) in the direction of $\vec{u}$"



Theorem

Let f be differentiable at the point (a,b). Then f has a directional derivative at (a,b) in the direction of $\hat{u}$. $\hat{u} = u_x\hat{i} + u_y\hat{j}$ and

$$\boxed{D_{\vec{u}} f(a,b) = \hat{u}\cdot\nabla f(a,b).}$$

this is used computationally

EX 1 Find the directional derivative of $f(x,y)$ at the point (a,b) in the direction of $\vec{u}$. (Note: $\vec{u}$ may not be a unit vector.)

a) $f(x,y) = y^2 \ln(x)$ $\quad (a,b) = (1,4)$ $\quad \vec{u} = \hat{i} - \hat{j}$ $\quad \|\vec{u}\| = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$\hat{u} = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$

$D_{\vec{u}} f(a,b) = \hat{u} \cdot \nabla f(a,b)$

$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle \frac{y^2}{x}, 2y \ln x \rangle$

$\nabla f(a,b) = \nabla f(1,4) = \langle \frac{4^2}{1}, 2(4)\ln 1 \rangle = \langle 16, 0 \rangle$

$D_{\hat{u}} f(a,b) = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle \cdot \langle 16, 0 \rangle = \frac{16}{\sqrt{2}} = 8\sqrt{2}$

b) $f(x,y) = 2x^2 \sin y + xy$ $\quad (a,b) = (1, \pi/2)$ $\quad \vec{u} = 2\hat{i} + \hat{j}$

$\nabla f(x,y) = \langle 4x \sin y + y, 2x^2 \cos y + x \rangle$

$\nabla f(1, \pi/2) = \langle 4 + \pi/2, 1 \rangle$

$\|\vec{u}\| = \sqrt{2^2+1^2} = \sqrt{5} \Rightarrow \hat{u} = \langle 2/\sqrt{5}, 1/\sqrt{5} \rangle$

$D_{\vec{u}} f(1, \pi/2) = \langle 4 + \pi/2, 1 \rangle \cdot \langle 2/\sqrt{5}, 1/\sqrt{5} \rangle$

$= \frac{8}{\sqrt{5}} + \frac{\pi}{\sqrt{5}} + \frac{1}{\sqrt{5}} = \frac{9+\pi}{\sqrt{5}}$

Maximum Rate of Change

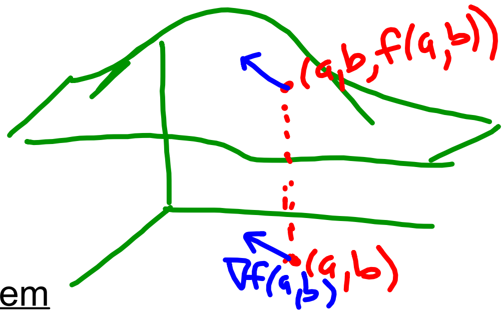
We know $D_{\vec{u}} f(a,b) = \hat{u} \cdot \nabla f(a,b)$

$$= \|\hat{u}\| \|\nabla f(a,b)\| \cos\theta = \|\nabla f(a,b)\| \cos\theta$$

What angle, θ, maximizes $D_{\vec{u}} f(a,b)$?

it's biggest when $\cos\theta = 1 \Leftrightarrow \theta = 0^\circ$

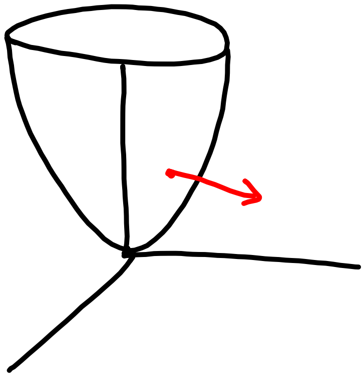
$\Rightarrow$ largest when $\vec{u}$ is in the direction of $\nabla f(a,b)$.



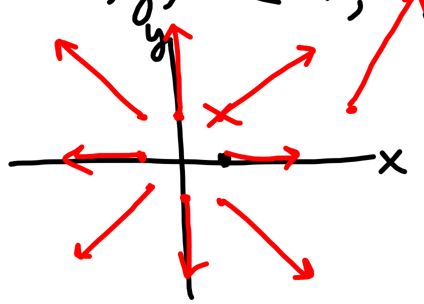
Theorem

The function, $z = f(x,y)$, increases most rapidly at (a,b) in the direction of the gradient (with rate $\|\nabla f(a,b)\|$) and decreases most rapidly in the opposite direction (with rate $-\|\nabla f(a,b)\|$).

EX 2 For $z = f(x,y) = x^2 + y^2$, interpret gradient vector.



$\nabla f(x,y) = \langle 2x, 2y \rangle$



EX 3 Find a vector indicating the direction of most rapid increase of $f(x,y)$ at the given point. Then find the rate of change in that direction.

a) $f(x,y) = e^y \sin x$ at $(a,b) = (5\pi/6, 0)$.

$\nabla f(x,y) = \langle e^y \cos x, e^y \sin x \rangle$

$\nabla f(5\pi/6, 0) = \langle \cos(5\pi/6), \sin(5\pi/6) \rangle = \langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

rate of change $= \|\nabla f(a,b)\| = \sqrt{(-\sqrt{3}/2)^2 + (1/2)^2} = \sqrt{1} = 1$

b) $f(x,y) = x^2y - 2/(xy)$ at $(a,b) = (1,1)$

$f(x,y) = x^2y - \frac{2}{xy}$

$\nabla f(x,y) = \langle 2xy - \frac{2}{y}\left(\frac{-1}{x^2}\right), x^2 - \frac{2}{x}\left(\frac{-1}{y^2}\right) \rangle$

$= \langle 2xy + \frac{2}{yx^2}, x^2 + \frac{2}{xy^2} \rangle$

$\nabla f(1,1) = \langle 4, 3 \rangle$ directn of steepest ascent (or max change)

$\|\nabla f(1,1)\| = \sqrt{16+9} = 5$

EX 4 The temperature at (x,y,z) of a ball centered at the origin is $T = 100e^{-(x^2+y^2+z^2)}$.

Show that the direction of greatest decrease in temperature is always a vector pointing away from the origin.

$$\nabla T(x,y,z) = \langle T_x, T_y, T_z \rangle$$

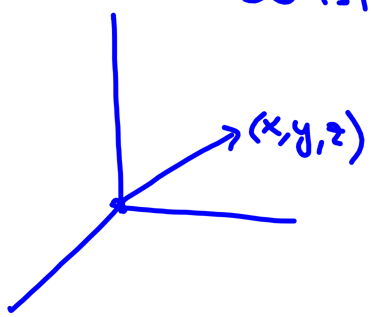
$$= \langle 100e^{-(x^2+y^2+z^2)}(-2x), 100e^{-(x^2+y^2+z^2)}(-2y), 100e^{-(x^2+y^2+z^2)}(-2z) \rangle$$

$$= -200e^{-(x^2+y^2+z^2)} \langle x, y, z \rangle$$

$\Rightarrow$ greatest decrease of T happens in this direction

$$200e^{-(x^2+y^2+z^2)} \langle x, y, z \rangle$$

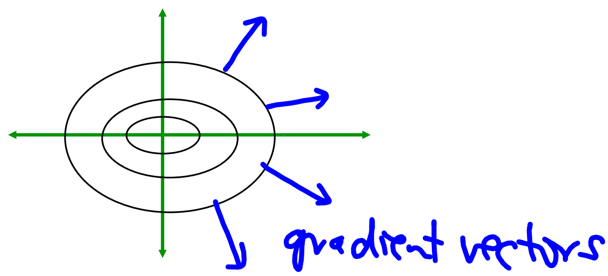
a vector pnting away from origin would be $\langle x,y,z \rangle$ or any positive constant multiple.



One extra (cool) fact

<u>Theorem</u>

The gradient of $z = f(x,y)$ $(w = f(x,y,z))$ at point P is perpendicular to the level curve (surface) of f through P.



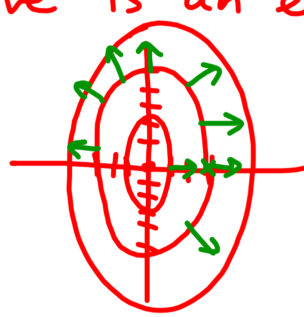
EX 5 Graph gradient vectors and level curves for

$$z = f(x,y) = \frac{x^2}{9} + \frac{y^2}{25} .$$

$\nabla f = \left\langle \frac{2x}{9}, \frac{2y}{25} \right\rangle$

each level curve is an ellipse.

(note: $z \geq 0$)



$\nabla f(3,0) = \left\langle \frac{6}{9}, 0 \right\rangle$

$\nabla f(0,5) = \left\langle 0, \frac{2}{5} \right\rangle$