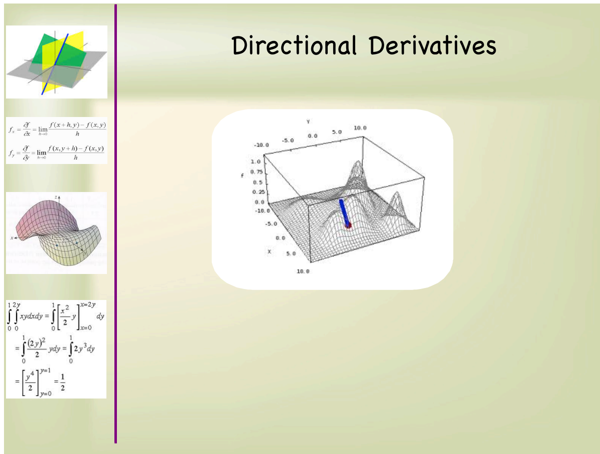


<u>Directional Derivatives</u>

We know we can write
$$\frac{\partial f}{\partial x} = f_x(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$$
$$\frac{\partial f}{\partial y} = f_y(a,b) = \lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}$$

The partial derivatives measure the rate of change of the function at a point <u>in the direction</u> of the x-axis or y-axis. What about the rates of change in the other directions?

<u>Definition</u>

For any unit vector, $\hat{u} = \langle u_x, u_y\rangle$ let
$$D_{\hat{u}} f(a,b) = \lim_{h\to 0}\frac{f(a+hu_x, b+hu_y)-f(a,b)}{h}$$
If this limit exists, this is called the directional derivative of f at the point (a,b) in the direction of $\hat{u}$.

<u>Theorem</u>

Let f be differentiable at the point (a,b). Then f has a directional derivative at (a,b) in the direction of $\hat{u}$. $\hat{u} = u_x\hat{\imath} + u_y\hat{\jmath}$ and $D_{\hat{u}}f(a,b) = \hat{u}\cdot\nabla f(a,b)$.

EX 1 Find the directional derivative of $f(x,y)$ at the point (a,b) in the direction of $\vec{u}$. (Note: $\vec{u}$ may not be a unit vector.)

a) $f(x,y) = y^2 \ln(x)$ $(a,b) = (1,4)$ $\vec{u} = \hat{i} - \hat{j}$

b) $f(x,y) = 2x^2 \sin y + xy$ $(a,b) = (1, \pi/2)$ $\vec{u} = 2\hat{i} + \hat{j}$

Maximum Rate of Change

We know $D_{\vec{u}} f(a,b) = \hat{u} \cdot \nabla f(a,b)$

$$= \|\hat{u}\| \|\nabla f(a,b)\| \cos\theta$$

What angle, θ, maximizes $D_{\vec{u}} f(a,b)$?

Theorem

The function, $z = f(x,y)$, increases most rapidly at (a,b) in the direction of the gradient (with rate $\|\nabla f(a,b)\|$) and decreases most rapidly in the opposite direction (with rate $-\|\nabla f(a,b)\|$).

EX 2 For $z = f(x,y) = x^2 + y^2$, interpret gradient vector.

EX 3 Find a vector indicating the direction of most rapid increase of $f(x,y)$ at the given point. Then find the rate of change in that direction.

a) $f(x,y) = e^y \sin x$ at $(a,b) = (5\pi/6,0)$.

b) $f(x,y) = x^2y - 2/(xy)$ at $(a,b) = (1,1)$

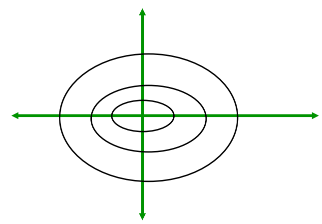
EX 4 The temperature at (x,y,z) of a ball centered at the origin is $T = 100e^{-(x^2+y^2+z^2)}$.

Show that the direction of greatest decrease in temperature is always a vector pointing away from the origin.

One extra (cool) fact

<u>Theorem</u>

The gradient of $z = f(x,y)$ $(w = f(x,y,z))$ at point P is perpendicular to the level curve (surface) of f through P.



EX 5 Graph gradient vectors and level curves for

$$z = f(x, y) = \frac{x^2}{9} + \frac{y^2}{25} \quad .$$