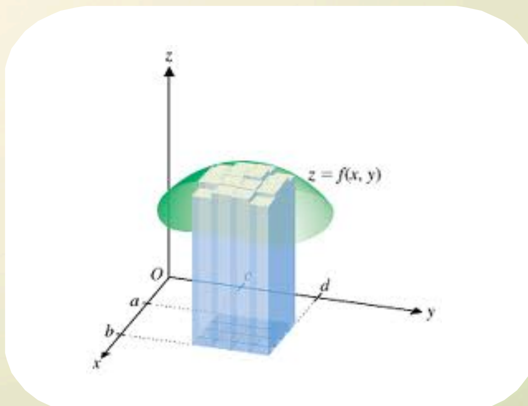
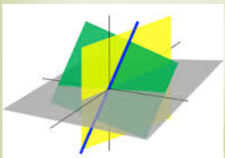
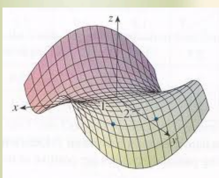
Double Integrals
Over
Non-rectangular Regions





$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$



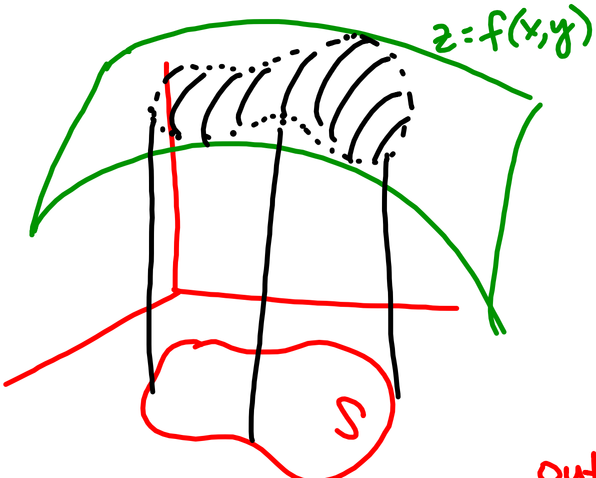
$$\int_0^1\int_0^{2y} xy\,dx\,dy = \int_0^1\left[\frac{x^2}{2}y\right]_{x=0}^{x=2y}dy$$

$$= \int_0^1\frac{(2y)^2}{2}y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

Double Integrals over Non-rectangular Regions

What if the region we're integrating over is not a rectangle, but a simple, closed curve region instead?



$$\boxed{V = \iint_S f(x,y)\,dA}$$

$$= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

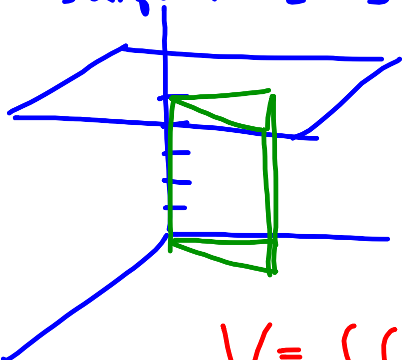
$$= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

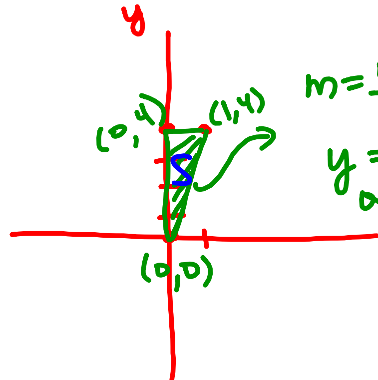
outer limits of integration are constants

limits of integration can be dependent on y

EX 1 Find $\iint_S 5\,dA$ where S is the triangle with vertices at $(0,0)$, $(0,4)$, and $(1,4)$.

Surface $z = 5$

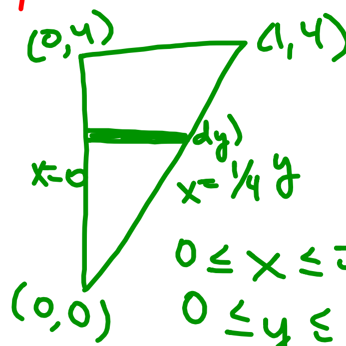




$m = \frac{4}{1} = 4$

$y = 4x$

or $x = \frac{1}{4}y$



$0 \le x \le \frac{1}{4}y$

$0 \le y \le 4$

$$V = \iint_S 5\,dx\,dy$$

$$= \int_0^4 \left[\int_0^{\frac{1}{4}y} 5\,dx\right] dy$$

$$= \int_0^4 \left(5x \Big|_0^{\frac{1}{4}y}\right) dy$$

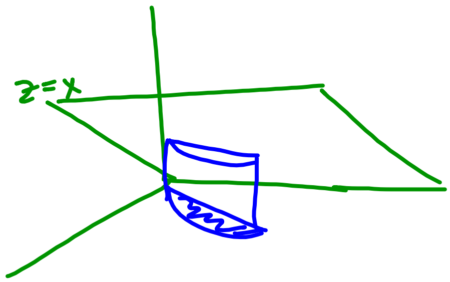
$$= \int_0^4 5\left(\frac{1}{4}y - 0\right) dy$$

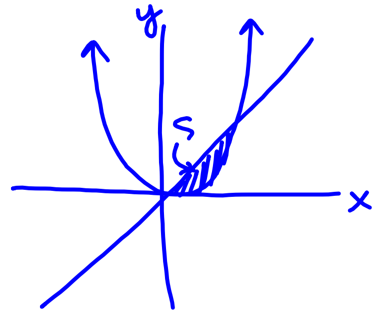
$$= \frac{5}{4}\int_0^4 y\,dy$$

$$= \frac{5}{4}\left(\frac{y^2}{2}\Big|_0^4\right)$$

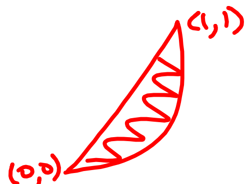
$$= \frac{5}{8}(4^2 - 0) = 5(2) = \boxed{10}$$

EX 2 Evaluate $\iint_S x\,dA$ where S is the region between $y = x$ and $y = x^2$ in the first octant.





$$V = \int_0^1 \int_{x^2}^{x} x \, dy\, dx$$



$$x^2 \le y \le x$$

$$0 \le x \le 1$$

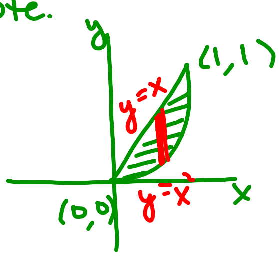
$$V = \int_0^1 x\left(y\Big|_{x^2}^{x}\right) dx$$

$$= \int_0^1 x(x - x^2)\,dx = \int_0^1 (x^2 - x^3)\,dx$$

$$= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

note:

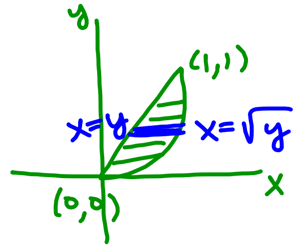
①



$$\text{Area} = \int_0^1 (x - x^2)\,dx$$

$$A = \int_0^1 \boxed{\int_{x^2}^{x} dy}\, dx$$

②



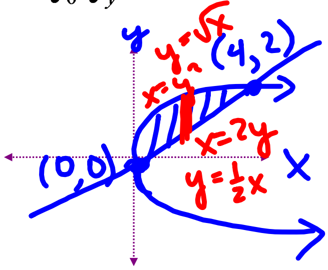
$$\text{Area} = \int_0^1 (\sqrt{y} - y)\,dy$$

$$A = \int_0^1 \boxed{\int_{y}^{\sqrt{y}} dx}\, dy$$

$$\int_{x^2}^{x} dy = y\Big|_{x^2}^{x} = x - x^2$$

EX 3 Write these integrals as iterated integrals with the order of integration switched.

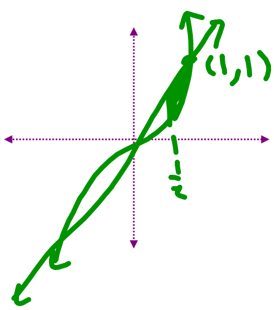
a) $\int_0^2 \int_{y^2}^{2y} f(x,y)\,dx\,dy = \int_0^4 \int_{\frac{1}{2}x}^{\sqrt{x}} f(x,y)\,dy\,dx$



① $y^2 \le x \le 2y$
② $0 \le y \le 2$

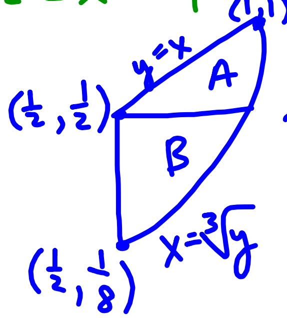
① $\frac{1}{2}x \le y \le \sqrt{x}$
② $0 \le x \le 4$

b) $\int_{1/2}^{1} \int_{x^3}^{x} f(x,y)\,dy\,dx = \int_{1/2}^{1} \int_{y}^{\sqrt[3]{y}} f(x,y)\,dx\,dy + \int_{\frac{1}{8}}^{\frac{1}{2}} \int_{\frac{1}{2}}^{\sqrt[3]{y}} f(x,y)\,dx\,dy$



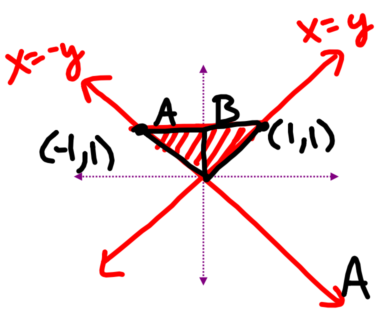
① $x^3 \le y \le x$
② $\frac{1}{2} \le x \le 1$

A $\begin{cases} ① & y \le x \le \sqrt[3]{y} \\ ② & \frac{1}{2} \le y \le 1 \end{cases}$



B $\begin{cases} ① & \frac{1}{2} \le x \le \sqrt[3]{y} \\ ② & \frac{1}{8} \le y \le \frac{1}{2} \end{cases}$

c) $\int_0^1 \int_{-y}^{y} f(x,y)\,dx\,dy = \int_{-1}^{0} \int_{-x}^{1} f(x,y)\,dy\,dx + \int_0^1 \int_x^1 f(x,y)\,dy\,dx$



① $-y \le x \le y$
② $0 \le y \le 1$

A $\begin{cases} ① & -x \le y \le 1 \\ ② & -1 \le x \le 0 \end{cases}$ B $\begin{cases} ① & x \le y \le 1 \\ ② & 0 \le x \le 1 \end{cases}$

EX 4 Evaluate

a) $\int_1^5 \int_0^x \frac{3}{x^2+y^2}\, dy\, dx$

remember

$$\int \frac{1}{a^2+u^2}\, du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + c$$

$$= \int_1^5 \left(\frac{3}{x}\arctan\left(\frac{y}{x}\right)\right)\Big|_0^x \, dx$$

$$= \int_1^5 \left(\frac{3}{x}\left[\arctan\left(\frac{x}{x}\right) - \arctan 0\right]\right) dx = \int_1^5 \frac{3\pi}{4}\left(\frac{1}{x}\right) dx$$

$$= \frac{3\pi}{4}\left(\ln|x|\right)\Big|_1^5 = \boxed{\frac{3\pi}{4}\ln 5}$$

b) $\int_{\pi/6}^{\pi/2} \int_0^{\sin\theta} 6r\cos\theta\, dr\, d\theta$

$$= \int_{\pi/6}^{\pi/2} 6\cos\theta\left(\frac{r^2}{2}\right)\Big|_0^{\sin\theta} d\theta$$

$$= \int_{\pi/6}^{\pi/2} 3\cos\theta\left(\sin^2\theta\right) d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$\theta = \pi/6,\ u = \sin(\pi/6) = \frac{1}{2}$

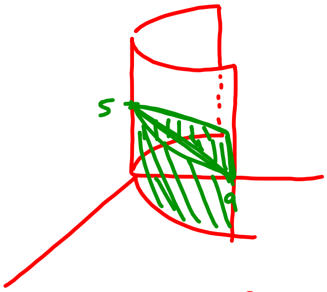
$\theta = \pi/2,\ u = \sin(\pi/2) = 1$

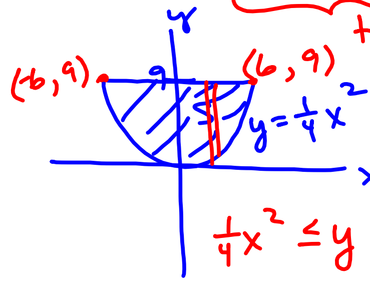
$$= \int_{1/2}^1 3u^2\, du$$

$$= u^3\Big|_{1/2}^1$$

$$= 1^3 - \left(\frac{1}{2}\right)^3 = \boxed{\frac{7}{8}}$$

EX 5 Find the volume of the solid bounded by the parabolic cylinder $x^2 = 4y$ and the planes $z = 0$ and $\underbrace{5y + 9z - 45 = 0}_{\text{top surface } z = 5 - \frac{5}{9}y}$.





$$\frac{1}{4}x^2 \le y \le 9$$
$$-6 \le x \le 6$$

$$V = \int_{-6}^{6} \int_{\frac{1}{4}x^2}^{9} \text{"top surface"}\, dy\, dx$$

$$= \int_{-6}^{6} \int_{\frac{1}{4}x^2}^{9} \left(5 - \frac{5}{9}y\right) dy\, dx$$

$$= \int_{-6}^{6} \left(5y - \frac{5}{18}y^2\right)\Big|_{\frac{1}{4}x^2}^{9} dx$$

$$= \int_{-6}^{6} \left[\left(45 - \frac{45}{2}\right) - \left(\frac{5}{4}x^2 - \frac{5}{18}\left(\frac{1}{16}x^4\right)\right)\right] dx$$

$$= \int_{-6}^{6} \left(\frac{45}{2} - \frac{5}{4}x^2 + \frac{5}{18(16)}x^4\right) dx$$

$$= \left(\frac{45}{2}x - \frac{5}{4}\left(\frac{x^3}{3}\right) + \frac{\cancel{5}}{18(16)}\left(\frac{x^5}{\cancel{5}}\right)\right)\Big|_{-6}^{6}$$

$$= \frac{45}{2}(6 - -6) - \frac{5}{12}\left(6^3 - (-6)^3\right) + \frac{6^5 - (-6)^5}{18(16)}$$

$$= 45(6) - 5(36) + \frac{6^4}{24}$$

$$= 270 - 180 + 54$$

$$= \boxed{144}$$