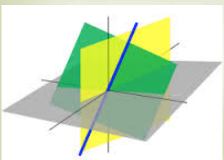
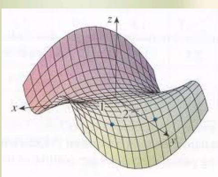


$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

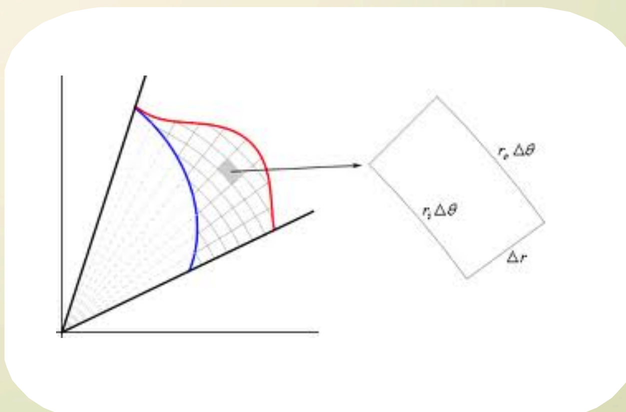


$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

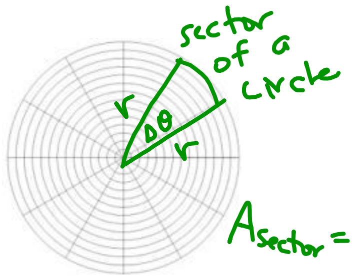
Double Integrals
in
Polar Coordinates



Double Integrals in Polar Coordinates

Rather than finding the volume over a rectangle (for Cartesian Coordinates), we will use a "polar rectangle" for polar coordinates.

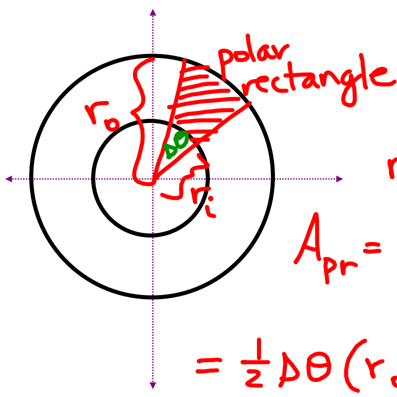
$dA = ?$



$$A_{sector} = \pi r^2 \left(\frac{\Delta\theta}{2\pi}\right) = \boxed{\frac{1}{2}\,\Delta\theta\, r^2}$$

$\Delta\theta =$ "a little bit of θ"

$A_{sector} = A_{circle}$ (fraction of circle that I want to keep) $= \pi r^2 \left(\frac{\Delta\theta}{2\pi}\right)$



Area of a polar rectangle

$r_o =$ outer radius, $r_i =$ inner radius

$$A_{pr} = A_{outer} - A_{inner}$$

$$= \frac{1}{2}\Delta\theta(r_o^2) - \frac{1}{2}\Delta\theta(r_i^2)$$

$$= \frac{1}{2}\Delta\theta(r_o^2 - r_i^2) = \frac{1}{2}\Delta\theta(r_o + r_i)(r_o - r_i)$$

$$A_{pr} = \Delta\theta\,\underbrace{(r_o - r_i)}_{\Delta r}\underbrace{\left(\frac{r_o + r_i}{2}\right)}_{\bar{r}} = \Delta\theta\,\Delta r\,\bar{r} = \underbrace{\bar{r}\,\Delta r\,\Delta\theta}$$

$\Rightarrow dA = r\,dr\,d\theta$

Then

$$V = \iint_S f(r,\theta)\, dA$$

$$= \iint_S f(r,\theta)\, r\, dr\, d\theta$$

or $\iint_S f(r,\theta)\, r\, d\theta\, dr$

remember

$$V \approx \underbrace{\sum_{k=1}^{n} \underbrace{f(\bar{r}_k, \bar{\theta}_k)}_{\text{ht of surface}}\, \underbrace{\bar{r}_k \Delta r_k \Delta\theta_k}_{\text{Area of polar rect.}}}_{\text{volume of that box over the polar rect. base}}$$

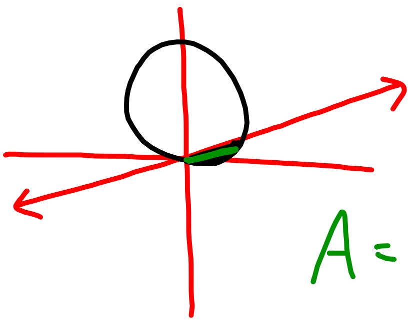
Why do we want to integrate polar coordinates?

because some integrals are doable/easier in polar coords.

EX 1 Find the area of the given region S by calculating

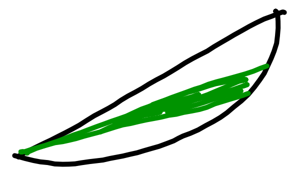
$$A = \iint_S dA = \iint_S r\, dr\, d\theta .$$

a) S is the smaller region bounded by $\theta = \pi/6$ and $r = 4\sin\theta$.



$0 \le r \le 4\sin\theta$

$0 \le \theta \le \pi/6$



$$A = \int_0^{\pi/6} \int_0^{4\sin\theta} r\, dr\, d\theta$$

$$A = \int_0^{\pi/6} \left(\frac{r^2}{2}\right)\Big|_0^{4\sin\theta} d\theta$$

$$= \int_0^{\pi/6} \frac{1}{2}(16\sin^2\theta)\, d\theta = 8\int_0^{\pi/6} \sin^2\theta\, d\theta$$

$$= \frac{8}{2}\int_0^{\pi/6} (1-\cos(2\theta))\, d\theta$$

$$= 4\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi/6}$$

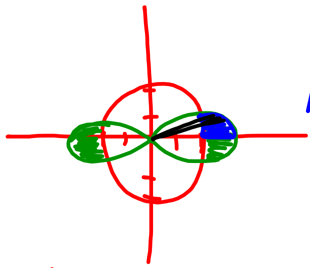
$$= 4\left(\frac{\pi}{6} - \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right) - 4(0-0) = \frac{2\pi}{3} - 2\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{2\pi}{3} - \sqrt{3}$$

EX 1 (cont'd) Find the area of the given region S by calculating

$$\iint_S r\, dr\, d\theta \ .$$

b) *S is the region outside the circle $r = 2$ and inside the lemniscate $r^2 = 9\cos(2\theta)$.*



$A = 4A_{blue}$

$A_{blue} = ?$

$2 \le r \le 3\sqrt{\cos(2\theta)}$

$0 \le \theta \le \frac{1}{2}\arccos\left(\frac{4}{9}\right)$

intersectn pt:

$r = 2 \quad r^2 = 9\cos(2\theta)$

$\frac{4}{9} = \cos(2\theta)$

$2\theta = \arccos\left(\frac{4}{9}\right)$

$\theta = \frac{1}{2}\cos^{-1}\left(\frac{4}{9}\right)$

$$A_{blue} = \int_0^{\frac{1}{2}\arccos\left(\frac{4}{9}\right)} \int_2^{3\sqrt{\cos(2\theta)}} r\, dr\, d\theta$$

$$= \int_0^{\frac{1}{2}\arccos\left(\frac{4}{9}\right)} \left(\frac{1}{2} r^2 \Big|_2^{3\sqrt{\cos(2\theta)}}\right) d\theta$$

$$= \int_0^{\frac{1}{2}\arccos\left(\frac{4}{9}\right)} \frac{1}{2}\left(9\cos(2\theta) - 4\right) d\theta$$

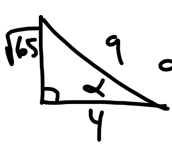
$$= \frac{9}{2}\int_0^{\frac{1}{2}\arccos\left(\frac{4}{9}\right)} \cos(2\theta)\, d\theta - 2\theta\Big|_0^{\frac{1}{2}\arccos\left(\frac{4}{9}\right)}$$

$$= \frac{9}{2}\left(\frac{1}{2}\right)\sin(2\theta)\Big|_0^{\frac{1}{2}\arccos\left(\frac{4}{9}\right)} - \arccos\left(\frac{4}{9}\right)$$

aside:

$\sin\left(2\left(\frac{1}{2}\arccos\left(\frac{4}{9}\right)\right)\right)$

$= \sin\left(\arccos\left(\frac{4}{9}\right)\right)$



$\alpha = \arccos\left(\frac{4}{9}\right)$

$\cos\alpha = \frac{4}{9}$

$\Rightarrow \sin\alpha = \frac{\sqrt{65}}{9}$

$$= \frac{9}{4}\left(\frac{\sqrt{65}}{9}\right) - \arccos\left(\frac{4}{9}\right)$$

$$= \boxed{\frac{\sqrt{65}}{4} - \arccos\left(\frac{4}{9}\right)}$$

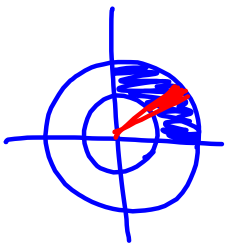
$$A = 4A_{blue} = 4\left(\frac{\sqrt{65}}{4} - \arccos\left(\frac{4}{9}\right)\right)$$

$$= \sqrt{65} - 4\arccos\left(\frac{4}{9}\right)$$

EX 2 Evaluate using polar coordinates.

a) $\iint\limits_S y\, dA$ where S is the first quadrant polar rectangle inside $x^2 + y^2 = 4$ and outside $x^2 + y^2 = 1$.

$r = 2$ $\qquad$ $r = 1$



$1 \le r \le 2$

$0 \le \theta \le \pi/2$

$$\iint\limits_S y\, dA = \int_0^{\pi/2}\int_1^2 (r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} (\sin\theta) \int_1^2 r^2\, dr\, d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left(\frac{r^3}{3}\Big|_1^2 \right) d\theta = \left(\frac{2^3}{3} - \frac{1^3}{3} \right) \int_0^{\pi/2} \sin\theta\, d\theta$$

$$= \frac{7}{3}\left(-\cos\theta\Big|_0^{\pi/2}\right) = -\frac{7}{3}\left(\cos\frac{\pi}{2} - \cos 0\right)$$

$$= -\frac{7}{3}(0-1) = \boxed{\frac{7}{3}}$$

b) $\iint\limits_S (x^2 + y^2)\, dA$

$$\int_0^{\pi/2}\int_1^2 (\, r^2 \,)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\left(\int_1^2 r^3\, dr\right) d\theta = \int_0^{\pi/2} \left(\frac{r^4}{4}\Big|_1^2 \right) d\theta$$

$$= \left(\frac{2^4}{4} - \frac{1^4}{4} \right)\left(\theta\Big|_0^{\pi/2} \right)$$

$$= \frac{15}{4}\left(\frac{\pi}{2}\right) = \boxed{\frac{15\pi}{8}}$$

EX 2 (cont'd) Evaluate using polar coordinates.

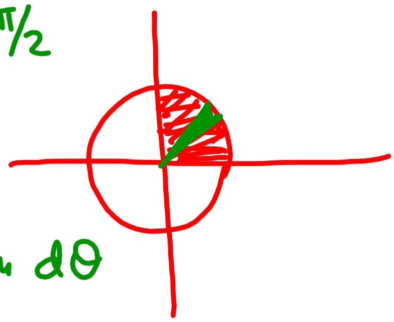
c) $\int_0^1 \int_0^{\sqrt{1-y^2}} \sin(x^2+y^2)\,dx\,dy$

$0 \le x \le \sqrt{1-y^2}$

$0 \le y \le 1$

$0 \le r \le 1$

$0 \le \theta \le \pi/2$



$x = \sqrt{1-y^2}$

$x^2+y^2=1$

$$= \int_0^{\pi/2} \int_0^1 \sin(r^2)\, r\,dr\,d\theta$$

$u = r^2$

$du = 2r\,dr$

$\frac{1}{2}du = r\,dr$

$r=0,\ u=0$

$r=1,\ u=1^2=1$

$$= \frac{1}{2}\int_0^{\pi/2}\int_0^1 \sin u\,du\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(-\cos u\Big|_0^1\right)d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}(\cos 1 - \cos 0)\,d\theta$$

$$= -\frac{1}{2}(\cos 1 - 1)\int_0^{\pi/2} d\theta = \frac{-1}{2}(\cos 1 - 1)\frac{\pi}{2}$$

$$= \frac{\pi}{4}(1-\cos 1)$$