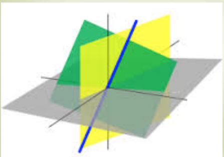
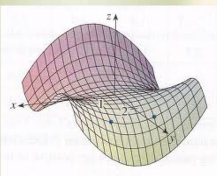


$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$



$$\int_0^1\int_0^{2y} xy\,dx\,dy = \int_0^1\left[\frac{x^2}{2}y\right]_{x=0}^{x=2y} dy = \int_0^1 \frac{(2y)^2}{2}y\,dy = \int_0^1 2y^3\,dy = \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

Triple Integrals

$$\begin{aligned}
\iiint_B 8xyz\,dV &= \int_1^2\int_2^3\int_0^1 8xyz\,dz\,dx\,dy \\
&= \int_1^2\int_2^3 4xyz^2\Big|_0^1\,dx\,dy \\
&= \int_1^2\int_2^3 4xy\,dx\,dy \\
&= \int_1^2 2x^2y\Big|_2^3\,dy \\
&= \int_1^2 10y\,dy = 15
\end{aligned}$$

Triple Integrals

$A = \int_a^b f(x)dx$ Measures 2-D space (signed area) under a curve above the x-axis.

$V = \iint_S f(x,y)dA$ Measures 3-D space (signed volume) under a surface above the xy-plane.

We predict that $\iiint_S f(x,y,z)dV$ measures 4-D space (signed) under a "hyper" surface "above" the xyz-"hyper-plane".

(ex think of $T = f(x,y,z)$ as temperature fn on S)

$$\boxed{\iiint_S f(x,y,z)dV = \int_{a_1}^{a_2} \int_{\varphi_1(x)}^{\varphi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\,dy\,dx}$$ (S is 3-d)

$\psi_1(x,y) \le z \le \psi_2(x,y)$, $\varphi_1(x) \le y \le \varphi_2(x)$, $a_1 \le x \le a_2$

Note: We can't draw anything in 4-D, but we can draw the region S in 3-D (domain space is now 3-D).

EX 1 Write an iterated integral for $\iiint\limits_S (y+z+1)\,dV$

where $S = \{(x,y,z) \mid x \in [0,1],\ y \in [2,5],\ z \in [1,4]\}$.

$dV = dx\,dy\,dz$

(or $dV = dy\,dx\,dz$ or any other order)

$$\iiint\limits_S (y+z+1)\,dx\,dy\,dz$$

$$= \int_1^4 \int_2^5 \int_0^1 (y+z+1)\,dx\,dy\,dz$$

$$= \int_1^4 \int_2^5 (y+z+1)x\Big|_0^1\,dy\,dz$$

$$= \int_1^4 \int_2^5 (y+z+1)\,dy\,dz$$

$$= \int_1^4 \left(\tfrac{1}{2}y^2 + (z+1)y\right)\Big|_2^5\,dz$$

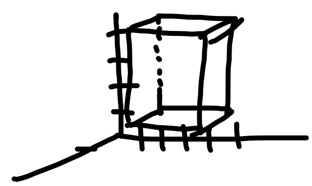
$$= \int_1^4 \left(\tfrac{25}{2} + 5z + 5 - 2 - 2z - 2\right)dz$$

$$= \int_1^4 \left(\tfrac{27}{2} + 3z\right)dz = \left(\tfrac{27}{2}z + \tfrac{3}{2}z^2\right)\Big|_1^4$$

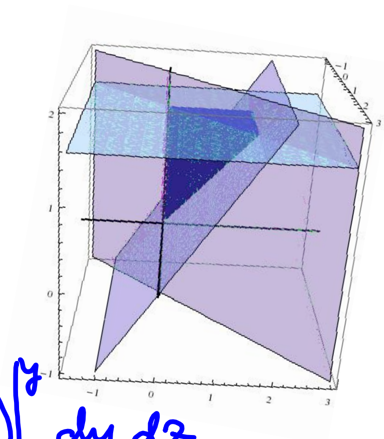
$$= \left(27(2) + 3(8)\right) - \left(\tfrac{27}{2} + \tfrac{3}{2}\right)$$

$$= 54 + 24 - 15 = \boxed{63}$$

domain space:



EX 2 Evaluate $\int_0^{\pi/2}\int_0^{z}\int_0^{y} \sin(x+y+z)\,dx\,dy\,dz$.



$$0 \le x \le y$$
$$0 \le y \le z$$
$$0 \le z \le \pi/2$$

$$= \int_0^{\pi/2}\int_0^{z} -\cos(x+y+z)\Big|_0^{y}\,dy\,dz$$

$$= \int_0^{\pi/2}\int_0^{z} \big(-\cos(2y+z)+\cos(y+z)\big)\,dy\,dz$$

$$= \int_0^{\pi/2} \left(-\frac{\sin(2y+z)}{2} + \sin(y+z)\right)\Big|_0^{z}\,dz$$

$$= \int_0^{\pi/2} \left[\left(-\frac{1}{2}\sin(3z)+\sin(2z)\right)-\left(-\frac{1}{2}\sin(z)+\sin(z)\right)\right]dz$$

$$= \int_0^{\pi/2} \left(-\frac{1}{2}\sin(3z)+\sin(2z)-\frac{1}{2}\sin z\right]dz$$

$$= \left(\frac{1}{6}\cos(3z)-\frac{1}{2}\cos(2z)+\frac{1}{2}\cos z\right)\Big|_0^{\pi/2}$$

$$= \left(\frac{1}{6}\underbrace{\cancel{\cos\left(\frac{3\pi}{2}\right)}}_{\to 0}-\frac{1}{2}\cos(\pi)+\frac{1}{2}\underbrace{\cancel{\cos\left(\frac{\pi}{2}\right)}}_{\to 0}\right)$$

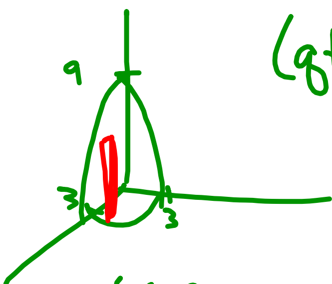
$$-\left(\frac{1}{6}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}\right) = -\frac{1}{2}(-1)-\frac{1}{6} = \boxed{\frac{1}{3}}$$

EX 3 Write an iterated integral for $\iiint_S f(x,y,z)\,dV$

where S is the region in the first octant bounded by the surface $z = 9 - x^2 - y^2$ and the coordinate planes.

S: $z = 9 - x^2 - y^2$, 1^{st} oct

(qtr of paraboloid)



$dV = dz\,dy\,dx$

$0 \le z \le 9 - x^2 - y^2$

$0 \le y \le \sqrt{9 - x^2}$

$0 \le x \le 3$

$$\iiint_S f(x,y,z)\,dV = \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} f(x,y,z)\,dz\,dy\,dx$$

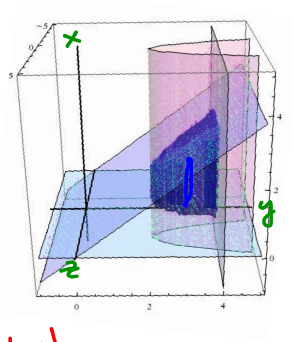
EX 4 Find the volume of the solid in the first octant bounded by the hyperbolic cylinder $y^2 - 64z^2 = 4$ and the planes $y = x$ and $y = 4$.

Note: $z = f(x,y)$

$$V = \iint_R f(x,y)\,dA \quad \text{(R region in 2d domain space)}$$

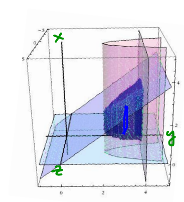
$$= \iiint_S 1\,dV$$

(S is 3-d region; solid)



ex $$V = \iint_R f(x,y)\,dx\,dy = \iint_R \left[\int_0^{f(x,y)} 1\,dz\right] dx\,dy$$

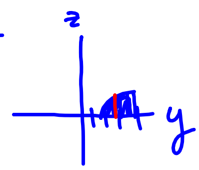
$$V = \iiint_S dV = \iiint_S dx\,dz\,dy$$



$0 \le x \le y$

$0 \le z \le \frac{1}{8}\sqrt{y^2-4}$

$2 \le y \le 4$



$y^2 - 64z^2 = 4$

$y = 4$

$y^2 - 4 = 64z^2$

$z = \frac{1}{8}\sqrt{y^2-4}$

$$V = \int_2^4 \int_0^{\frac{1}{8}\sqrt{y^2-4}} \int_0^y dx\,dz\,dy$$

$$= \int_2^4 \int_0^{\frac{1}{8}\sqrt{y^2-4}} y\,dz\,dy$$

$$= \int_2^4 \frac{1}{8} y\sqrt{y^2-4}\,dy$$

$u = y^2 - 4$

$du = 2y\,dy$

$\frac{1}{2}du = y\,dy$

$y = 2,\ u = 2^2 - 4 = 0$

$y = 4,\ u = 4^2 - 4 = 12$

$$= \int_0^{12} \frac{1}{8}\left(\frac{1}{2}\right)\sqrt{u}\,du$$

$$= \frac{1}{16}\left(\frac{2}{3}u^{3/2}\right)\Big|_0^{12}$$

$$= \frac{1}{24}\left(12^{3/2} - 0\right) = \frac{1}{24}\left(2\sqrt{3}\right)^3$$

$$= \frac{8(3)\sqrt{3}}{24} = \boxed{\sqrt{3}} \text{ units}^3$$

EX 5 Find the volume of the tetrahedron with vertices at (0,0,0), (0,0,3), (0,4,0), and (2,0,0).

plane ("roof")

$$ax+by+cz=d$$

$$3c=d,\quad 4b=d,\quad 2a=d$$

choose $d=12$, $c=4$, $b=3$, $a=6$

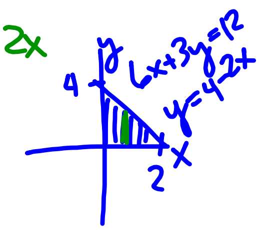
$$\Rightarrow 6x+3y+4z=12$$

$$V=\iiint_S dV=\iiint_S dz\,dy\,dx$$

$0\le z\le 3-\frac{3}{4}y-\frac{3}{2}x$

$0\le y\le 4-2x$

$0\le x\le 2$



$$=\int_0^2\int_0^{4-2x}\int_0^{3-\frac{3}{4}y-\frac{3}{2}x} dz\,dy\,dx$$

$$=\int_0^2\int_0^{4-2x}\left(3-\frac{3}{4}y-\frac{3}{2}x\right)dy\,dx$$

$$=\int_0^2\left(3y-\frac{3}{8}y^2-\frac{3}{2}xy\right)\Big|_0^{4-2x}dx$$

$$=\int_0^2\left(3(4-2x)-\frac{3}{8}(4-2x)^2-\frac{3}{2}x(4-2x)\right)dx$$

$$=\int_0^2\left(12-\cancel{6x}-\frac{3}{8}(16-\cancel{16x}+4x^2)-6x+3x^2\right)dx$$

$$=\int_0^2\left(6-6x+\frac{3}{2}x^2\right)dx$$

$$=\left(6x-3x^2+\frac{1}{2}x^3\right)\Big|_0^2=\left(\cancel{12}-3(\cancel{4})+\frac{1}{2}(8)\right)-0$$

$$=\boxed{4\text{ units}^3}$$