Math 2210 #28
Independence of Path

Recall the Fundamental Theorem of Calculus.

$$\int_a^b f'(x)dx = f(b) - f(a)$$

We would like an analogous theorem for line integrals.

Fundamental Theorem of Line Integrals

Let C be the curve given by the parameterization $\vec{r}(t), t \in [a, b]$, such that $\vec{r}(t)$ is differentiable. If $f(\vec{r})$ is continuously differentiable on an open set containing C, then

$$\int_C \nabla f(\vec{r}) \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

EX 1

Find work done by ∇ along a curve going from

$$(1,1,1) \text{ to } (4,-1,2), \text{ given } f(r) = \frac{c}{\|\vec{r}\|} \nabla f = \frac{-c\vec{r}}{\|\vec{r}\|}$$

A set, D, is called a <u>Path-Connected Set</u> if any 2 points in D can be joined by a piece-wise smooth curve lying entirely in D.

Example Non-example

What does it mean to be independent of path?

Independence of Path Theorem

Let $\vec{F}(\vec{r})$ be continuous on an open connected set D.
Then $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ is independent of any path, C, in D iff $\vec{F}(\vec{r}) = \nabla f(\vec{r})$ for some $f(\vec{r})$ (scalar function),
i.e. if $\vec{F}(\vec{r})$ is a conservative vector field on D.

Equivalent Conditions for Line Integrals

Let $\vec{F}(\vec{r})$ be continuous on an open connected set D.
The following statements are equivalent.

a) $\vec{F} = \nabla f$ for some f (i.e. $\vec{F}$ is conservative on D).

b) $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ is independent of the path, C, in D.

c) $\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = 0$ for every closed path in D.

Theorem

Let $\vec{F} = M\hat{\imath} + N\hat{\jmath} + P\hat{k}$ with M, N, P continuously differentiable on a ball, D.

Then $\vec{F}$ is conservative $\Leftrightarrow \nabla \times \vec{F} = \vec{0}$.

Note:

If $\vec{F} = M\hat{\imath} + N\hat{\jmath}$

then $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix} = \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

and

$$\nabla \times \vec{F} = \vec{0} \Rightarrow \frac{dN}{dx} = \frac{dM}{dy}$$

EX 2

Is $\vec{F} = (12x^2 + 3y^2 + 5y)\hat{\imath} + (6xy - 3y^2 + 5x)\hat{\jmath}$ conservative?

EX 3

Using $\vec{F}$ from Example 1, find f such that $\vec{F} = \nabla f$.

EX 4

Using $\vec{F} = (12x^2 + 3y^2 + 5y)\hat{\imath} + (6xy - 3y^2 + 5x)\hat{\jmath}$
calculate $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ where C is any path from (0,0) to (2,1).

EX 5

Show that the line integral $\int_C ((yz+1)dx + (xz+1)dy + (xy+1)dz)$ is independent of path and evaluate the integral, where C is a curve from $(0,1,0)$ to $(1,1,1)$.

EX 6

Let $\vec{F} = \left(1 + 2xy\sin\left(x^2y\right)\right)\hat{\imath} + \left(1 + x^2\sin\left(x^2y\right)\right)\hat{\jmath}$

Is $\vec{F}$ conservative?

If yes, then find f such that $\vec{F} = \nabla f$.

EX 7

Evaluate $\int_{(0,0)}^{(1,\pi/2)} (e^x \sin y dx + e^x \cos y dy)$.