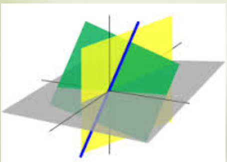
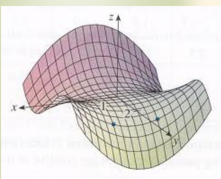


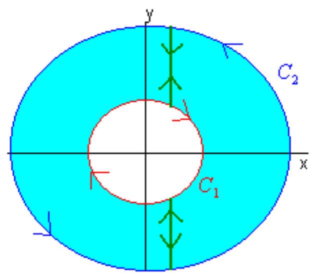
$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy$$
$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$
$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

# Green's Theorem



## Goal:

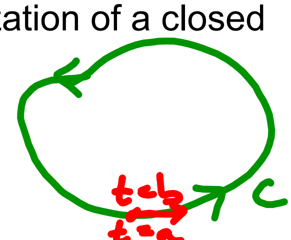
Describe the relation between the way a fluid flows along or across the boundary of a plane region and the way fluid moves around inside the region.

## Circulation or flow integral

Assume $\vec{F}(x,y)$ is the velocity vector field of a fluid flow. At each point $(x,y)$ on the plane, $\vec{F}(x,y)$ is a vector that tells how fast and in what direction the fluid is moving at the point $(x,y)$.

Assume $\vec{r}(t)=x(t)\hat{\imath} + y(t)\hat{\jmath},\ t\in[a,b]$, is parameterization of a closed curve lying in the region of fluid flow.

($\vec{r}$ has implied direction, start at $t=a$, end at $t=b$)



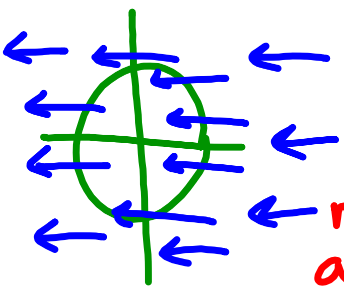

Let $\vec{F}(x,y) = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$ .

We want to measure "how much" fluid is moving along the curve $\vec{r}(t)$.

$\vec{F}(x(t),y(t)) \cdot \vec{r}\,'(t)$ measures the flow along the curve, C (given by parameterization $\vec{r}(t)$), at the pt $(x(t),y(t))$.

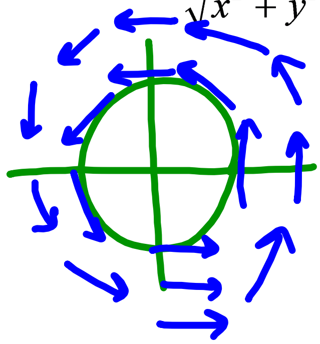
EX 1 Let $\vec{r}(t)$ be the parameterization of the unit circle centered at the origin. Draw these vector fields and think about how the fluid moves around that circle.

(a) $\vec{F}(x,y) = -2\hat{i}$



some (not many) are moving along the circle

(b) $\vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2 + y^2}}$



all vectors touching circle are moving along the circle

When $\vec{F}(x,y)$ is parallel to the tangent line at a point, then the maximum flow is along a circle.

When $\vec{F}(x,y)$ is perpendicular to the tangent line at a point, then there is no flow along the circle.

So $\vec{F}(x,y)\cdot\vec{T}(x,y)$ measures the flow along the circle where $\vec{T}(x,y)=\vec{r}\,'(t)$ .

(tangent vector to curve C)

We define the circulation of $\vec{F}$ along $C$, a parameterized curve from $t = a$ to $t = b$ as

$$\int_a^b \vec{F}(x,y)\cdot\vec{r}\,'(t)dt = \int_a^b \vec{F}\cdot d\vec{r} = \int_{t=a}^{t=b} Mdx + Ndy$$

EX 2 Given $C$: $x = a\cos t$, $t \in [0, 2\pi]$

$y = a\sin t$,

find the circulation along $C$ for each of these.

a) $\vec{F}_1(x,y) = 2\hat{i}$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{t=a}^{t=b} M\,dx + N\,dy$$

we have

$M = 2$, $N = 0$

$dx = -a\sin t\,dt$

$$= \int_0^{2\pi} (2(-a\sin t))\,dt$$

$$= 2a\cos t\Big|_0^{2\pi}$$

$$= 2a(1-1) = 0$$

b) $\vec{F}_2(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$

$M = \dfrac{-y}{\sqrt{x^2+y^2}}$, $N = \dfrac{x}{\sqrt{x^2+y^2}}$

$dx = -a\sin t\,dt$

$dy = a\cos t\,dt$

$M = \dfrac{-a\sin t}{\sqrt{a^2}} = -\sin t$

$N = \dfrac{a\cos t}{\sqrt{a^2}} = \cos t$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{t=a}^{t=b} M\,dx + N\,dy$$

$$= \int_0^{2\pi} (-\sin t)(-a\sin t)\,dt + \cos t(a\cos t)\,dt$$

$$= \int_0^{2\pi} a\,dt = a\left(t\Big|_0^{2\pi}\right)$$

$$= 2\pi a$$

Flux across a curve

Given $\vec{F}(x,y) = M\hat{i} + N\hat{j}$ (vector velocity field) and a curve $C$, with the parameterization $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, $t \in [a,b]$, such that $C$ is a positively oriented, simple, closed curve.

We want to know the rate at which a fluid is entering and leaving the area of the region enclosed by a curve, $C$. This is called flux.

$\vec{F}(x,y) \cdot \vec{n}(x,y)$ is the component of $\vec{F}$ perpendicular to the curve,

so flux $= \boxed{\oint_c \vec{F}\cdot\vec{n}\, ds}$

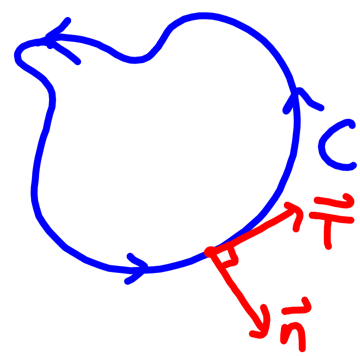
($\vec{n}$ is normal vector to curve $C$)

Now to find $\vec{n} = \vec{T} \times \hat{k}$

(unit vector)

$$= \left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}\right) \times \hat{k}$$

$$= \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}$$




This means

$$\vec{F} \cdot \vec{n} = M\frac{dy}{ds} - N\frac{dx}{ds}$$

flux $= \displaystyle\oint_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds$

$$= \boxed{\oint_C M dy - N dx}$$

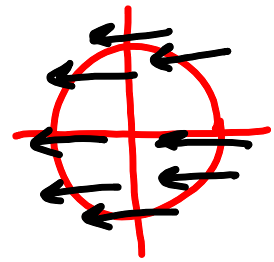
EX 3 Find the flux across $C$: $\vec r(t) = \overbrace{(a\cos t)}^{x(t)}\hat i + \overbrace{(a\sin t)}^{y(t)}\hat j,\ t \in [0,2\pi]$

$$\oint_C \vec F\cdot\vec n\, ds = \oint_C M\,dy - N\,dx$$

$dx = -a\sin t\,dt$

$dy = a\cos t\,dt$

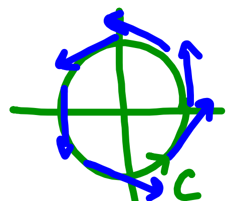
a) $\vec F_1(x,y) = -2\hat i$



$M=-2,\ N=0$

$$\text{flux} = \int_0^{2\pi} -2(a\cos t)\,dt = -2a\left(-\sin t\Big|_0^{2\pi}\right) = 0$$

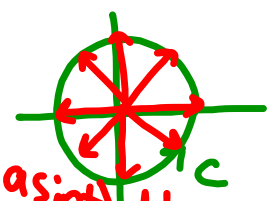
b) $\vec F_2(x,y) = \dfrac{-y\hat i + x\hat j}{\sqrt{x^2+y^2}} = \underbrace{(-\sin t)}_{M}\hat i + \underbrace{(\cos t)}_{N}\hat j$




$$\text{flux} = \int_0^{2\pi}(-\sin t)(a\cos t\,dt) - (\cos t)(-a\sin t)\,dt$$

$$= \int_0^{2\pi}(-a\sin t\cos t + a\sin t\cos t)\,dt = 0$$

c) $\vec F_3(x,y) = x\hat i + y\hat j = \underbrace{(a\cos t)}_{M}\hat i + \underbrace{(a\sin t)}_{N}\hat j$




$$\text{flux} = \int_0^{2\pi}(a\cos t)(a\cos t)\,dt - a\sin t(-a\sin t)\,dt$$

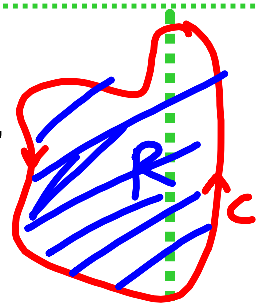
$$= \int_0^{2\pi}(a^2\cos^2 t + a^2\sin^2 t)\,dt$$

$$= a^2\int_0^{2\pi} 1\,dt = a^2\left(t\Big|_0^{2\pi}\right) = \boxed{2\pi a^2}$$

# Two Forms of Green's Theorem in The Plane

Let $\vec{F}(x,y) = M\hat{i} + N\hat{j}$

Let $C$ be a simple, closed, positively oriented curve enclosing a region $R$ in the $xy$-plane.

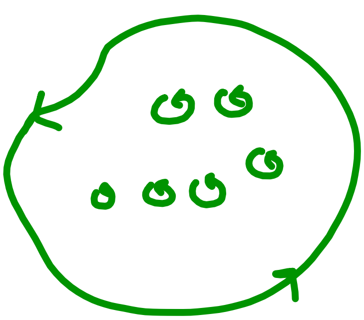



$$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

$$\oint_C \vec{F}\cdot\vec{n}\;ds = \iint_R \nabla\cdot\vec{F}\;dA$$

(flux across the boundary of C)

$\nabla\cdot\vec{F} = \text{div}\,\vec{F}$



Let $\vec{F}(x,y) = M\hat{i} + N\hat{j}$

Let $C$ be a simple, closed, positively oriented curve enclosing a region $R$ in the $xy$-plane.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\;dy$$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot\vec{T}\;dt = \iint_R \nabla\times\vec{F}\cdot\hat{k}\;dA$$

(circulation along boundary curve C)

$(\nabla\times\vec{F})\cdot\hat{k}$ measures the "microscopic" circulation of the vector field at pt $(x,y)$ in $R$

$\nabla\times\vec{F}$ measures how much the vector field circulates at a pt

$(\nabla\times\vec{F})\cdot\hat{k}$ gives magnitude of that circulation

EX 5 Verify both forms of Green's theorem for the field

$\vec{F}(x,y) = (x-y)\hat{i} + x\hat{j}$

and the region $R$ bounded by the circle

$C: \vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j},\ t \in [0,2\pi]$

$M = x-y,\ N = x$

$M_x = 1 \quad N_x = 1$

$M_y = -1 \quad N_y = 0$

$dx = -\sin t\, dt$

$dy = \cos t\, dt$

$M = \cos t - \sin t$

$N = \cos t$

① flux

$$\oint_C (M\,dy - N\,dx)$$

$$= \int_0^{2\pi} (\cos t - \sin t)(\cos t)\,dt - \cos t(-\sin t)\,dt$$

$$= \int_0^{2\pi} \cos^2 t\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} (1+\cos(2t))\,dt$$

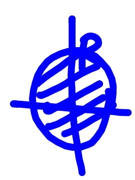
$$= \frac{1}{2}\left(t + \frac{1}{2}\sin(2t)\right)\Big|_0^{2\pi}$$

$$= \frac{1}{2}(2\pi) = \pi$$

check:

$$\iint_R (M_x + N_y)\,dx\,dy$$

$$= \iint_R 1\,dx\,dy$$





$$= \int_0^{2\pi}\int_0^1 r\,dr\,d\theta$$

$$= 2\pi\left(\frac{r^2}{2}\right)\Big|_0^1 = \pi \checkmark$$

② circulation

$$\oint_C (M\,dx + N\,dy)$$

$$= \int_0^{2\pi} (1 - \sin t\cos t)\,dt$$

$$= \left(t + \frac{1}{4}\cos(2t)\right)\Big|_0^{2\pi}$$

$$= 2\pi$$

check:

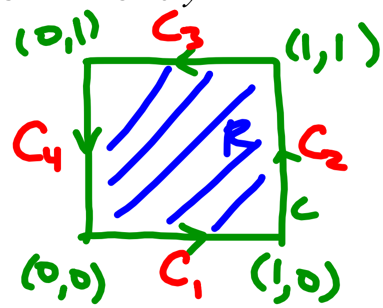
$$\iint_R (N_x - M_y)\,dx\,dy$$

$$= \iint_R (1+1)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 2r\,dr\,d\theta$$

$$= 2\pi \checkmark$$

EX 6 Evaluate the integral $\oint_C (xy\,dy - y^2\,dx)$ where $C$ is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.




$$\oint_C (xy\,dy - y^2\,dx)$$

$$= \oint_C M\,dy - N\,dx$$

$M = xy$, $N = y^2$

$C = C_1 \cup C_2 \cup C_3 \cup C_4$

$$\oint_C M\,dy - N\,dx = \iint_R (M_x + N_y)\,dx\,dy \qquad \text{(flux)}$$

(by Green's Thm)

$$\iint_R (y + 2y)\,dx\,dy = \int_0^1 \int_0^1 (3y)\,dx\,dy$$

$$= \int_0^1 3y \left(x\Big|_0^1\right) dy$$

$$= 3\int_0^1 y\,dy$$

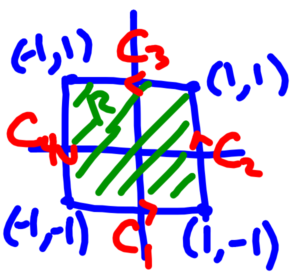
$$= 3\left(\frac{y^2}{2}\Big|_0^1\right) = \frac{3}{2}$$

EX 7 Calculate the flux of the field $\vec{F}(x,y) = x\hat{i} + y\hat{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

flux

$$\oint_C M\,dy - N\,dx = \iint_R (M_x + N_y)\,dA$$

(by Green's Thm)




$M = x, \quad N = y, \quad M_x = 1, \quad N_y = 1$

$$\iint_R (M_x + N_y)\,dA$$

$$= \int_{-1}^{1}\int_{-1}^{1} 2\,dA = \int_{-1}^{1}\int_{-1}^{1} 2\,dx\,dy$$

$$= \int_{-1}^{1} \left(2x\Big|_{-1}^{1}\right) dy = 4y\Big|_{-1}^{1} = 8$$

How do we use Green's Thm to create another way to find area of region R?

we know $A = \iint_R dA$

choose M and N such that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$

because Green's Thm says

$$\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \oint_C M\,dx + N\,dy$$

if $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$, we get

$$A = \iint_R dA = \oint_C M\,dx + N\,dy$$

Let $N = \frac{1}{2}x$, $M = \frac{-1}{2}y$

(check: $N_x - M_y = \frac{1}{2} - \frac{-1}{2} = 1$) ✓

$$\Rightarrow A = \iint_R dA = \boxed{\oint_C \frac{-1}{2}y\,dx + \frac{1}{2}x\,dy}$$