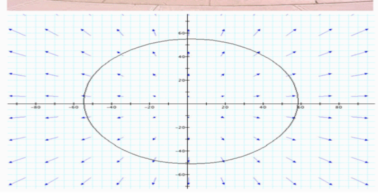
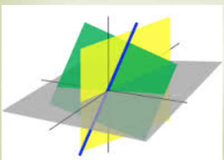
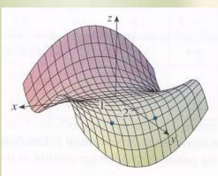
Gauss's Divergence Theorem





$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

Gauss's Divergence Theorem

Let $\vec{F}(x,y,z)$ be a vector field continuously differentiable in the solid, S.

S a 3-D solid
∂S the boundary of S (a surface)
$\hat{n}$ unit outer normal to the surface ∂S
$div\,\vec{F}$ divergence of $\vec{F}$

Then $\underbrace{\iint_{\partial S} \vec{F}(x,y,z)\cdot\vec{n}\,dS}_{\text{Surface integral}} = \iiint_{S} div\vec{F}\,dV$ (3-d)

dV = "a little bit of volume" = dx dy dz

This is the 3-d version of the flux application of Green's Thm!

remember:

(2-d) $\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R div\,\vec{F}\,dA$

The rate of flow through a boundary of S $= \iint\limits_{\partial S} \vec{F}(x,y,z)\cdot\hat{n}\, dS$

If there is net flow out of the closed surface, the integral is positive.
If there is net flow into the closed surface, the integral is negative.
This integral is called "flux of $\vec{F}$ across a surface ∂S". $\vec{F}$ can be any vector field, not necessarily a velocity field.

Gauss's Divergence Theorem tells us that the flux of $\vec{F}$ across ∂S can be found by integrating the divergence of $\vec{F}$ over the region enclosed by ∂S.

EX 1 $\vec{F}(x,y,z) = x^3\hat{i}+y^3\hat{j}+z^3\hat{k}$

S is the hemisphere $0 < z < \sqrt{a^2 - x^2 - y^2}$ (hemisphere)

Calculate $\iint\limits_{\partial S} \vec{F}\cdot\vec{n}\, dS.$ (flux across hemisphere)

we know $\iint\limits_{\partial S} \vec{F}\cdot\vec{n}\, dS = \iiint\limits_{S} \text{div}\,\vec{F}\, dV$

$$\nabla\cdot\vec{F} = \text{div}\,\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x^3\hat{i} + y^3\hat{j} + z^3\hat{k}\right)$$

$$= 3x^2 + 3y^2 + 3z^2$$

$$\iiint\limits_{S} 3(x^2+y^2+z^2)\, dV$$ (switch to spherical coords)

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} 3\rho^2(\rho^2 \sin\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= 3(2\pi)\int_0^{\pi/2} \sin\varphi \int_0^{a} \rho^4\, d\rho\, d\varphi$$

$$= 6\pi \int_0^{\pi/2} \sin\varphi\left(\frac{a^5}{5}\right) d\varphi$$

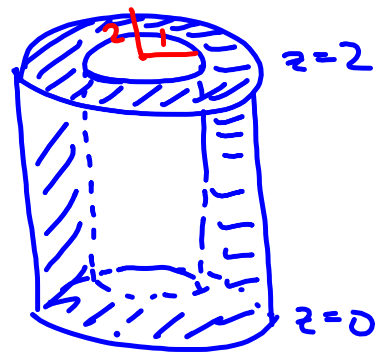
$$= \frac{6\pi a^5}{5}\left(-\cos\varphi \Big|_0^{\pi/2}\right) = \frac{-6\pi a^5}{5}(0-1)$$

$$= \frac{6\pi a^5}{5}$$

EX 2 $\vec{F}(x,y,z) = 27\hat{i}+x\hat{j}+z^2\hat{k}$

S is the solid cylindrical shell $1 \le x^2+y^2 \le 4,\ 0 \le z \le 2$

Calculate $\iint_{\partial S} \vec{F}\cdot\vec{n}\, dS.$



$$\iint_{\partial S} \vec{F}\cdot\vec{n}\, dS = \iiint_S \operatorname{div}\vec{F}\, dV$$

$$\operatorname{div}\vec{F} = 0+0+2z = 2z$$

use (cylindrical coords)

$$\Rightarrow \iiint_S \operatorname{div}\vec{F}\, dV = \int_0^{2\pi}\int_0^2\int_1^2 (2z)\, r\, dr\, dz\, d\theta$$

$$= 2\pi(2)\int_0^2\int_1^2 z r\, dr\, dz$$

$$= 4\pi \int_0^2 z\left(\tfrac{1}{2}r^2\Big|_1^2\right) dz$$

$$= 2\pi \int_0^2 (4-1)\, z\, dz$$

$$= 6\pi\left(\frac{z^2}{2}\Big|_0^2\right)$$

$$= 3\pi(4-0) = \boxed{12\pi}$$

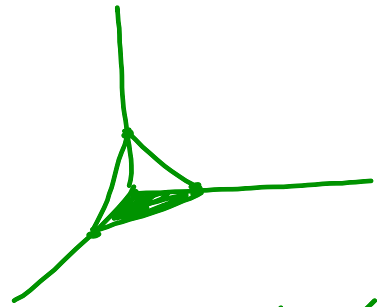
EX 3 $\vec{F}(x,y,z) = x\hat{i}+y\hat{j}+z\hat{k}$

S is the solid enclosed by $x+y+z=1,\ x=0,\ y=0,\ z=0$

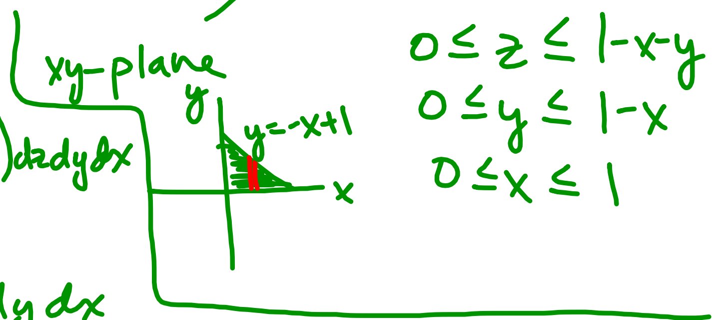
Calculate $\iint_{\partial S} \vec{F}\cdot\vec{n}\,dS.$

$$\iint_{\partial S} \vec{F}\cdot\vec{n}\,dS$$

$$= \iiint_S \operatorname{div}\vec{F}\,dV$$



$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y}(1+1+1)\,dz\,dy\,dx$$



$0 \le z \le 1-x-y$

$0 \le y \le 1-x$

$0 \le x \le 1$

$$= \int_0^1\int_0^{1-x}\left(3z\Big|_0^{1-x-y}\right)dy\,dx$$

$$= 3\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$$

$$= 3\int_0^1\left(y-xy-\frac{y^2}{2}\right)\Big|_0^{1-x}dx$$

$$= 3\int_0^1\left(1-x-x(1-x)-\frac{1}{2}(1-x)^2\right)dx$$

$$= 3\int_0^1\left(1-x-\cancel{x}+x^2-\frac{1}{2}+\cancel{x}-\frac{1}{2}x^2\right)dx$$

$$= 3\int_0^1\left(\frac{1}{2}-x+\frac{1}{2}x^2\right)dx$$

$$= 3\left(\frac{1}{2}x-\frac{x^2}{2}+\frac{1}{6}x^3\right)\Big|_0^1$$

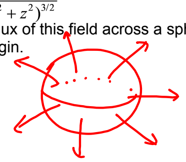
$$= 3\left[\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right)-0\right] = \frac{1}{2}$$

EX 4 Define $\vec{E}(x,y,z)$ to be the electric field created by a point-charge, q located at the origin.

$$\vec{E}(x,y,z) = q\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

(pointing away from origin)

Find the outward flux of this field across a sphere of radius a centered at the origin.



We cannot apply the divergence theorem to a sphere of radius a around the origin because our vector field is NOT continuous at the origin.

Applying it to a region between two spheres, we see that

$$\text{Flux} = \iiint\limits_{\substack{annular\\region}} div\vec{E}\, dV = 0$$

(applying Gauss' Div. Thm)

annular region

because $div\,\vec{E} = 0$. (prove to yourself) $\nabla\cdot\vec{E} = 0$

The field entering from the sphere of radius a is all leaving from sphere b,

$$\text{so } \boxed{\iint\limits_{\substack{sphere\\r=a}} \vec{F}\cdot\vec{n}\,dS = \iint\limits_{\substack{sphere\\r=b}} \vec{F}\cdot\vec{n}\,dS}$$

To find flux: directly evaluate

$$\iint\limits_{sphere} \vec{F}\cdot\hat{n}\,dS = \iint\limits_{sphere} \underbrace{q\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2+y^2+z^2)^{3/2}}}_{\vec{E}} \cdot \overbrace{\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{\sqrt{(x^2+y^2+z^2)}}}^{\text{unit outward normal for a sphere}}\,dS$$

$$= \iint\limits_{sphere} q\left(\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2}\right)dS$$

$$= q\iint\limits_{sphere} \frac{1}{(x^2+y^2+a^2-x^2-y^2)}\,dS$$

$$= \frac{q}{a^2}\iint\limits_{sphere} dS$$

remember: $z^2 = a^2 - x^2 - y^2$ (sphere)

$$dS = \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy \quad \text{where } z = f(x,y)$$

$$z = \pm\sqrt{a^2-x^2-y^2}$$

$$dS = \sqrt{\frac{x^2+y^2}{a^2-x^2-y^2}+1}\,dx\,dy$$

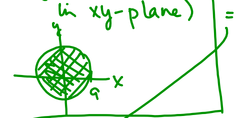
$$dS = \frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

$$f_x = \frac{\pm x}{\sqrt{a^2-x^2-y^2}}$$

$$f_y = \frac{\pm y}{\sqrt{a^2-x^2-y^2}}$$

$$\Rightarrow \iint\limits_{sphere} \vec{F}\cdot\vec{n}\,dS = \frac{q}{a^2}\iint\limits_{R} \frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

R: circle (projectn of sphere in xy-plane)



(switch to polar coords)

$$= \frac{q}{a^2}\int_0^{2\pi}\int_0^a \frac{a}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta$$

$u = a^2 - r^2$

$du = -2r\,dr$

$-\frac{1}{2}du = r\,dr$

$r=0,\ u=a^2$

$r=a,\ u=0$

$$= \frac{q(a)}{a^2}\int_0^{2\pi}\int_{a^2}^{0} -\frac{1}{2}\left(\frac{1}{\sqrt{u}}\right)du\,d\theta$$

$$= -\frac{q}{2a}(2\pi)\int_{a^2}^{0} u^{-1/2}\,du$$

$$= -\frac{\pi q}{a}\left(2\sqrt{u}\,\Big|_{a^2}^{0}\right)$$

$$= -\frac{\pi q}{a}(2(0-a)) = \boxed{2\pi q}$$

(answer does NOT depend on radius of sphere)