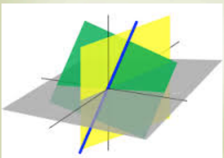
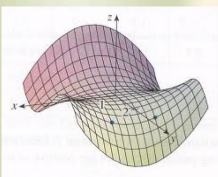
Stokes's Theorem



$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

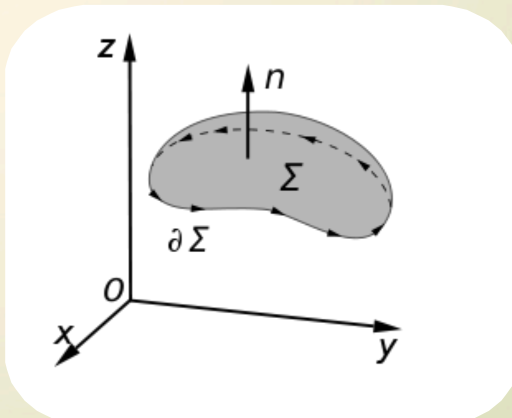
$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$



Remember this form of Green's Theorem:

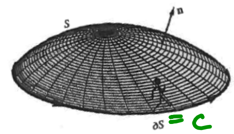
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \nabla\times\vec{F}\cdot\hat{k}\, dA$$

where $\vec{F}(x,y) = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$,

C is a simple closed positively-oriented curve that encloses a closed region, R, in the xy-plane.

It measures circulation along the boundary curve, C.

Stokes's Theorem generalizes this theorem to more interesting surfaces.



Stokes's Theorem

For $\vec{F}(x,y,z) = M(x,y,z)\hat{\imath}+N(x,y,z)\hat{\jmath}+P(x,y,z)\hat{k}$,

M, N, P have continuous first-order partial derivatives.
S is a 2-sided surface with continuously varying unit normal, $\hat{n}$,
C is a piece-wise smooth, simple closed curve, positively-oriented that is the boundary of S,
$\hat{T}$ is the unit tangent vector to C,

then $$\oint_C \vec{F}\cdot\hat{T}\, ds = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, dS = \iint_R (\nabla\times\vec{F})\cdot\vec{n}\, dx\,dy$$

Note: Assume S is given by $z=f(x,y)$.
Then remember that $\vec{n} = \langle f_x, f_y, -1\rangle$.

Also remember that dS (a little bit of surface) is given by

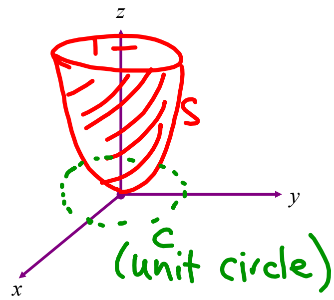
$$dS = \sqrt{f_x^2+f_y^2+1}\, dx\,dy$$

$$\Rightarrow \hat{n} = \frac{\langle f_x, f_y, -1\rangle}{\sqrt{f_x^2+f_y^2+1}}$$

$$\Rightarrow \hat{n}\, dS = \vec{n}\, dx\,dy$$

$$\Rightarrow \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, dS$$
$$= \iint_R (\nabla\times\vec{F})\cdot\hat{n}\sqrt{f_x^2+f_y^2+1}\, dx\,dy$$
$$= \iint_R (\nabla\times\vec{F})\cdot\vec{n}\, dx\,dy$$

EX 1 Verify Stokes's Theorem for $\vec{F} = y^2\hat{i} - x\hat{j} + 5z\hat{k}$ if S is the paraboloid $z = x^2 + y^2$ with the circle $x^2 + y^2 = 1$ as its boundary.
$(z=1)$



Ⓐ (surface integral)

$$\nabla \times \vec{F} = \text{curl}\,\vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -x & 5z \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-1-2y)$$

$$= (-1-2y)\hat{k}$$

$$\vec{n} = \langle f_x, f_y, -1 \rangle \text{ or } \langle -f_x, -f_y, 1 \rangle$$

$$= \langle -2x, -2y, 1 \rangle$$

$$(\nabla \times \vec{F}) \cdot \vec{n} = 0(-2x) + 0(-2y) + (-1-2y)(1) = -1-2y$$

$$\Rightarrow \iint_R (\nabla \times \vec{F}) \cdot \vec{n}\, dx\, dy = \iint_R (-1-2y)\, dx\, dy$$

(R is region inside C; a unit circle)

switch to polar coords:
$x = r\cos\theta \quad y = r\sin\theta$
$r \in [0,1], \ \theta \in [0, 2\pi]$

$$= \int_0^{2\pi} \int_0^1 (-1 - 2r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(-\frac{r^2}{2} - 2\sin\theta\left(\frac{r^3}{3}\right) \right) \Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \left(-\frac{1}{2} - \frac{2}{3}\sin\theta \right) d\theta$$

$$= \left(-\frac{1}{2}\theta + \frac{2}{3}\cos\theta \right) \Big|_0^{2\pi} = -\frac{1}{2}(2\pi - 0) + \frac{2}{3}\underbrace{(\cos(2\pi) - \cos 0)}_{0}$$

$$= -\pi$$

(B) (line integral)

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \int_C M\,dx + N\,dy + P\,dz$$

$$= \int_C y^2\,dx - x\,dy + 3z\,dz$$

C: unit circle, again it's easier to switch to polar coords.

$x = \cos\theta \qquad y = \sin\theta \qquad z = 1$

$dx = -\sin\theta\,d\theta \qquad dy = \cos\theta\,d\theta \qquad dz = 0$

$$= \int_0^{2\pi} (\sin^2\theta)(-\sin\theta)\,d\theta - \cos^2\theta\,d\theta + 0$$

$$= \int_0^{2\pi} (-\sin^3\theta - \cos^2\theta)\,d\theta$$

$$= \underbrace{\int_0^{2\pi} (-\sin^3\theta)\,d\theta}_{0} - \int_0^{2\pi} \cos^2\theta\,d\theta$$

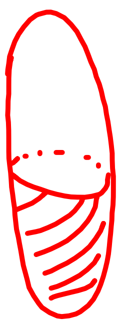
$$= -\frac{1}{2}\int_0^{2\pi} (1+\cos(2\theta))\,d\theta$$

$$= -\frac{1}{2}\left(\theta + \underbrace{\frac{\sin(2\theta)}{2}}_{0}\right)\Big|_0^{2\pi}$$

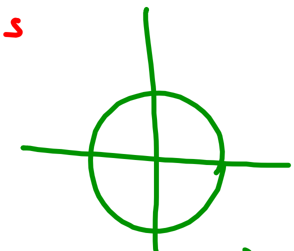
$$= -\frac{1}{2}(2\pi - 0) = -\pi \quad \checkmark$$

EX 2 Use Stokes's Theorem to calculate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$
for $\vec{F} = \overset{M}{xz^2}\hat{i} + \overset{N}{x^3}\hat{j} + \overset{P}{\cos(xz)}\hat{k}$
where S is the part of the ellipsoid $\boxed{x^2 + y^2 + 3z^2 = 1}$
below the xy-plane and $\hat{n}$ is the lower normal.

$C \quad (z=0)$



$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \oint_C \vec{F} \cdot \vec{T}\, ds$$



$x^2 + y^2 = 1$

$$= \int_C M\,dx + N\,dy + P\,dz$$

$$= \int_C xz^2\,dx + x^3\,dy + \cos(xz)\,dz$$

use polar coords $\quad x = \cos\theta \quad y = \sin\theta$

$z = 0 \qquad dx = -\sin\theta\, d\theta \quad dy = \cos\theta\, d\theta$

$dz = 0$

$$= \int_0^{2\pi} \cos\theta(0)(-\sin\theta\, d\theta) + \cos^3\theta(\cos\theta)\,d\theta + 0$$

$$= \int_0^{2\pi} \cos^4\theta\, d\theta = \frac{-3\pi}{4}$$

EX 3 Let S be a solid sphere. Show that $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = 0$

a) by using Stokes's Theorem

b) by using Gauss's Theorem

(a) Let S be sphere $x^2+y^2+z^2=r^2$ (r fixed)

Split S into $S_1 \cup S_2$ where S_1 is top half of sphere and S_2 is bottom half of sphere.

Then $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$

$$= \iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{n}\, dS + \iint_{S_2} (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

(by Stokes's Thm) $= \oint_C \vec{F} \cdot \vec{T}\, ds + \oint_{-C} \vec{F} \cdot \vec{T}\, ds$ (C is circle $x^2+y^2=r^2$)

$= 0$

Stokes's Thm

essentially says that the curl over the surface (from the vector field) is the same as the rotation along edges

Note:

Green's Thm is special case of Stokes's Thm

(b) we can show $\operatorname{div}(\operatorname{curl} \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0 \ \forall \vec{F}$.

And Gauss' Thm says

$$\iint_S \vec{F} \cdot \hat{n}\, dS = \iiint_{\text{solid}} \operatorname{div} \vec{F}\, dV$$

$$\Rightarrow \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \iiint_{\text{solid}} \operatorname{div}(\nabla \times \vec{F})\, dV$$

$= 0$