Intro to Scientific Computing: Solutions

Dr. David M. Goulet

1. How many steps does it take to separate 32 objects into groups of 4?

   We start with 2^5 objects and apply 3 steps of the algorithm to reduce the pile to groups of $2^{5-3} = 4$.

2. How many steps does it take to separate 23 objects into groups of less than 3?

   Note that $2^4 < 23 < 2^5$, and in 3 steps of the algorithm, a group of 2^4 wold be reduced to groups of $2^{4-3} = 2$. Therefor, using the last problem, it requires 3 steps.

3. If your haystack is originally a 10 meter cube, and you can't see the needle until the stack is a 1 centimeter cube, how many steps will the bisection method require?

   Note that 1m^3 is $(100^3\text{cm}^3$, so we need to split $10,000,000$ into groups of 1 to be sure that we've found the needle. $\log_2 10^7 \approx 23.25$. Therefor, we require 24 steps.

4. If you have a set of spheres, cubes and pyramids each in three different colors, how many steps of the algorithm will guarantee that they are sorted into groups of the same geometry and color?

   So split the colors into distinct groups you need 2 steps. Now that each group is of similar color, you need 2 more steps to separate each into different geometries. Another way to see this is to note that there will be 9 groups at the end, so it will require 4 steps because $2^3 < 9 < 2^4$.

5. A king is placed somewhere on a chess board but you are not allowed to look at the board. You are allowed to pick a vertex and this vertex is defined as the origin of a coordinate system. You are then told which

quadrant the king lies in. How many iterations of this "quadsection" method will guarantee that you find the king.

There are 64 total squares. We are essentially being asked to split these into groups of 1. So it requires $\log^4 64 = 3$ steps.

1. Use the theorem to show that $f(x) = x^2 - 1$ has a root somewhere in the interval $[0, 3]$.

   $f(0) = -1$ and $f(3) = 8$

2. Use the theorem to show that $f(x) = e^x - (x + 2)\cos(x)$ has a root somewhere in the interval $[0, \pi/2]$.

   $f(0) = -2$ and $f(\pi/2) = e^{\pi/2} > 0$.

3. If a runner finishes a 10 mile race in 60 minutes, was there a 1 mile section that the runner ran in exactly 6 minutes?

   Let the runner's distance traveled by time t be given by $f(t)$. Now, define $g(t) = f(t) - f(t - 6)$. Note that $g(0) = 0$. Suppose that $g(t) < 1$ for all t. Then

$$\begin{aligned} 10 &= f(60) - f(0) \\ &= f(60) - f(54) + f(54) - f(48) + \ldots + f(6) - f(0) \\ &= g(60) + g(54) + \ldots + g(6) < 10 \end{aligned}$$

   This is a contradiction. So there is at least one value of $t \in [0, 60]$ so that $g(t) \geq 1$. Because $g(0) = 0$, the intermediate value theorem tells us that there is at least one value of t so that $g(t) = 1$.

1. If $f(0) = -1$ and $f(8) = 1$, how many steps of the bisection method will be required to find an approximation to the root of $f(x)$ accurate to 0.25?

   The width of the initial interval is 8. After 4 steps, the interval will have width $8 * 2^{-4} = 0.5$. So, choosing our guess to be the midpoint of this interval, the error can be no more than 0.25.

2. You've built a machine that shoots darts at a target. By aiming it at an angle $\theta = \pi/3$, your first dart lands 10cm above the center of the bulls eye. You adjust the angle to $\theta = \pi/4$, and your second dart lands

10cm below the center of the bulls eye. Can the bisection method be used to find the correct angle? Can we say with certainty how many steps will be required?

Our range is initially 21 units wide and our domain is initially $\pi/12$ units wide. In theory, we could keep bisecting domains until we've narrowed the range to a 1 unit space around the center of the target.

If there were a linear relationship between θ and y, the height at which the dart strikes, then we could apply 1 step of the bisection method and hit the target in the center, using $\theta = (\pi/3 + \pi/4)/2 = 7\pi/24$.

Placing the dart machine at the origin of a coordinate system, and using Newtonian physics (ignoring friction), gives an equation for the height at which the dart strikes.

$$y = d \tan\theta - \frac{g\, d^2}{2} \sec^2\theta$$

Here d is the x-distance of the target from the machine and g is the acceleration of gravity. So, there is a non-linear relationship between the input, θ, and the output y. So we can not easily conclude how many steps will be required.

1. Approximate a root of the polynomial $x^5 - 3x^4 - 6x^3 + 18x^2 + 8x - 24$ with an accuracy of at least 0.0001. From this, guess what the exact value is and then use this information to factor the polynomial. Then apply the algorithm again to find another root. Can you find all five roots this way? (Note: I know that there are other tricks for root finding, don't use them.)

   The integer roots of this equation are $\{\pm 2, 3\}$. They should be easy with the bisection method after plotting the function. Using polynomial or synthetic division, we find $\frac{x^5-3x^4-6x^3+18x^2+8x-24}{(x-2)(x+2)(x-3)} = x^2 - 2$. This is easy to factor.

2. Obviously $f(x) = x^4$ has a root at $x = 0$. Can the bisection method approximate this root?

   $f(x) > 0$ for $x \neq 0$, so the bisection method is not useful because we need an interval on which the function changes sign.

3. The bisection method does not help when searching for the roots of $f(x) = x^2 + 1$. Why? Can you think of a way to modify the bisection method to find roots of $f(x)$?

   The roots are not real because $x^2 + 1 \geq 1$ for real x. Letting $x = \imath y$ gives $1 - y^2$ which has real roots that can be found with the bisection method.

1. Does this method appear to converge for $f(x) = x^2 - 1$, $x_0 = 3$, and $x_1 = 2$?

   Newton's method takes the form $x_{n+1} = x_n - \frac{(x_n^2-1)(x_n-x_{n-1})}{x_n^2-x_{n-1}^2}$. The plot indicates convergence.

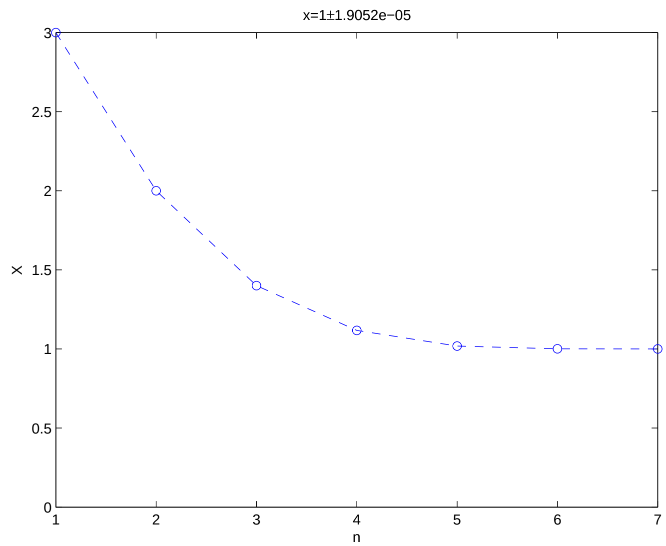

Figure 1: Newton's method for $x^2 - 1$.

2. Can you think of a way to apply these ideas to higher dimensional problems? Apply your idea to finding the root of $f(x, y) = x^2 + y^2$.

   First pick two guesses, (x_0, y_0) and (x_1, y_1). Then create the line

$$
\begin{aligned}
x &= (x_0 - x_1)t + x_1 \\
y &= (y_0 - y_1)t + y_1 \\
z &= (f(x_0, y_0) - f(x_1, y_1))t + f(x_1, y_1)
\end{aligned}
$$

This crosses the $x-y$ plane at $t=\frac{-f(x_1,y_1)}{f(x_0,y_0)-f(x_1,y_1)}$. This motivates the numerical scheme

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n,y_n)(x_n-x_{n-1})}{f(x_n,y_n)-f(x_{n-1},y_{n-1})} \\ y_{n+1} &= y_n - \frac{f(x_n,y_n)(y_n-y_{n-1})}{f(x_n,y_n)-f(x_{n-1},y_{n-1})} \end{aligned}$$

1. Does this method appear to converge for $f(x)=x^2-\sin^2 x$, $x_0=-1$, and $x_1=2$? The plot indicates convergence.

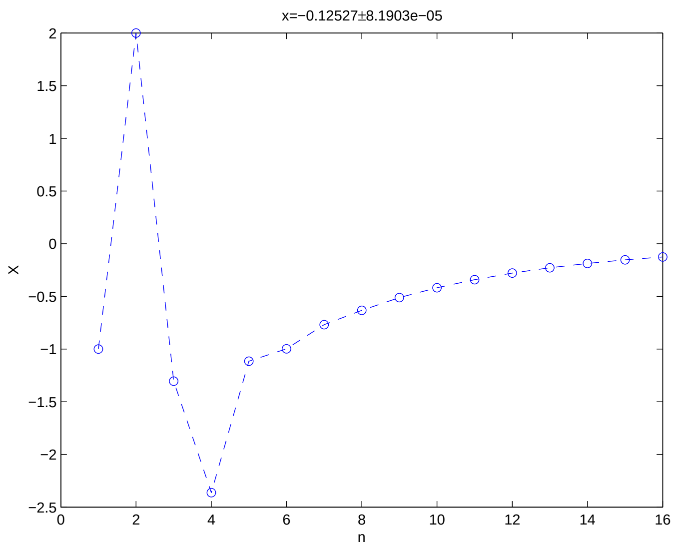

Figure 2: Newton's method for $x^2-\sin^2 x$.

2. Does this method appear to converge for $f(x)=e^{-x^{-2}}$, $x_0=-1$, and $x_1=2$? What about $x_0=1$ and $x_1=2$? In both cases, the iterates do not converge.

1. How can you combine these two ideas together with Newton's method to find the correct value of ω? Solving for ω^2 in the second equation and inserting this into the first gives

$$\int_0^\theta \frac{ds}{\sqrt{\cos(s)-\cos(\theta)}} - \tau\sqrt{2g/l} = 0$$

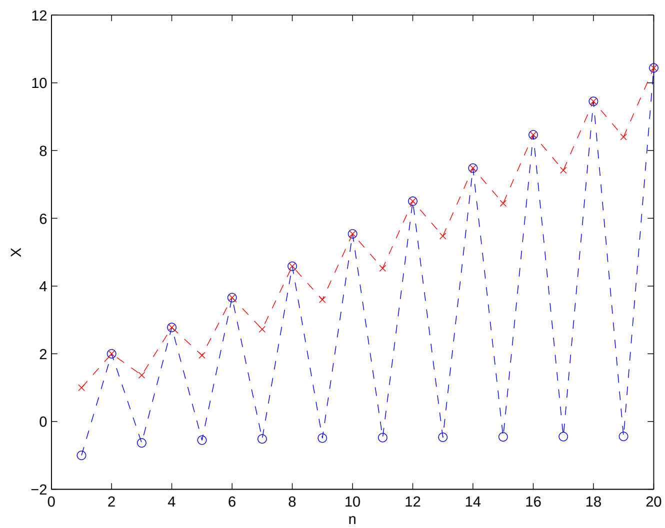

Figure 3: Newton's method for $e^{-x^{-2}}$. Blue o's show the first initial data iterates, red x's show the second

The left hand side can be defined as $g(\theta)$. If Newton's method solves $g(\theta) = 0$ then we use this value of θ to calculate

$$\omega = \sqrt{(2g/l)(1 - \cos\theta)}$$

2. Given the plot of $f(x) = \int_0^x \frac{ds}{\sqrt{\cos(s) - \cos(x)}}$, do you think Newton's method will converge? It may help to know that $f(0^+) = \frac{149}{60\sqrt{2}} \approx 1.756$.

   From the plot, we see that if $\tau\sqrt{2g/l} < \frac{149}{60\sqrt{2}}$ then there is no solution to the problem. Otherwise this looks like the problem of finding when a parabola-like thing intersects the x-axis, which our experience tells us may be doable with Newton's method.

Be warned that the way this is coded, you can't choose values for x outside of $(0, \pi)$. Do you think that this will matter when using Newton's method?

If we choose some valid value for $\tau\sqrt{2g/l}$ and pick our initial guesses in the range $(0, \pi)$, it appears from the graph that our subsequent guesses will stay in this domain. So it may not be a problem. Below we show how this intuition is validated for $\tau\sqrt{2g/l} = 3$, $x_0 = 2$, and $x_1 = 2$.

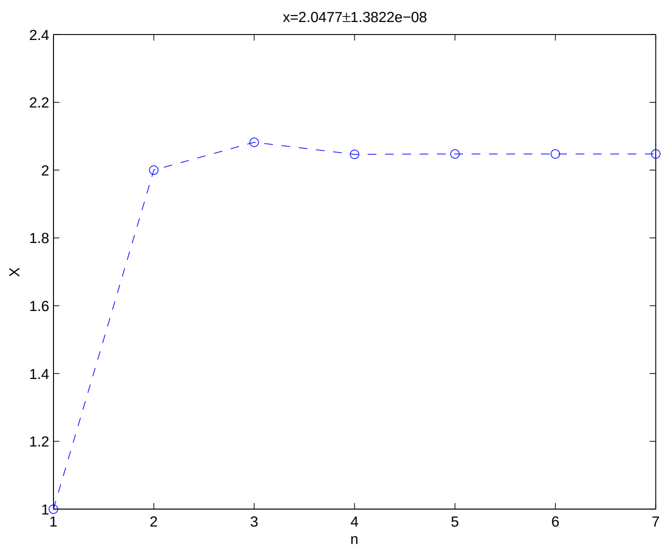

Figure 4: Convergence of Newton's method for the pendulum problem.