

FAIR CHAOTIC AMOEBA INVENTIONS^{1 2}

Peter Trapa
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Begin at the beginning and go on till you come to the end: then stop.

— *Lewis Carroll*

Consider a linear array of 10 boxes (a so-called “10-board”):



Suppose you have a bag that contains an endless supply of 1-tiles, i.e. square tiles of the form



and an endless collection of 2-tiles of the form



Problem 1. How many ways can one tile the 10-board by 1- and 2-tiles? What about the 15-board?

¹The title is an anagram (which I will reveal on the next page). Revealing it now would give too much away.

²An excellent reference for this material (and much more) is the book *Proofs That Really Count* by Arthur Benjamin and Jennifer Quinn published by the Mathematical Association of America. (At the time of writing, a significant portion of the book was freely available from books.google.com .) My notes are shamelessly stolen from notes by Hugo Rossi available at <http://www.math.utah.edu/teacherscircle/schedule.html> .

It isn't difficult to argue directly that

$$c(j, n) = \binom{n}{j}$$

where, by definition,

$$\binom{n}{j} = \frac{n!}{j!(n-j)!} = \binom{n}{n-j}$$

if $0 \leq j \leq n$ and zero otherwise. But we can also reason as follows. Clearly $\binom{n}{n} = 1$ for all n , and we can do a little algebra to check that

$$\binom{n}{j} = \binom{n}{j-1} + \binom{n-1}{j-1}.$$

Thus we immediately get that $c(j, n) = \binom{n}{j}$.

Notice that we can think of $c(j, n) = \binom{n}{j}$ as the number of ways of choosing j colored tiles out of the set of n boxes. This is probably the way that you have seen $\binom{n}{j}$ before. It counts the way of choosing j elements out of a set of size n .

Let's return to the original tiling problem and ask a related question:

Problem 3. How many tilings of the n board use exactly j 2-tiles?

We can reason as follows. If we use exactly j tiles, then we have to use exactly $n - 2j$ 1-tiles, and so $n - j$ tiles in all. Of those $n - j$ tiles we need to select j of them to be 2-tiles. So, indeed,

$$\text{the number of tilings of the } n\text{-board with } j \text{ 2-tiles is } \binom{n-j}{j}.$$

Now since f_n counts the number of all tilings of the n -board,

$$\begin{aligned} f_n = & \text{the number of tilings of the } n\text{-board with 0 2-tiles +} \\ & \text{the number of tilings of the } n\text{-board with 1 2-tiles +} \\ & \text{the number of tilings of the } n\text{-board with 3 2-tiles + } \cdots . \end{aligned}$$

So we get the remarkable identity

$$f_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \cdots .$$

What does this look like in terms of Pascal's Triangle? For instance, take $n = 6$. We get that $f_6 = 13$ is the sum of the starred numbers below:

$$\begin{array}{cccccccc} & & & & & & & 1 \\ & & & & & & & 1 & 1 \\ & & & & & & 1 & 2 & 1 \\ & & & & 1 & 3 & 3 & 1 & 1* \\ & & 1 & 4 & 6* & 4 & & & \\ & 1 & 5* & 10 & 10 & & & & \\ 1* & 6 & 15 & 20 & & & & & \end{array}$$

More generally, we start with a left-most 1 in the n th row of Pascal's triangle and make "knight jumps" moving up one row and over two entries until we run out of room. The sum of the entries visited in this way is the n th Fibonacci number. Pretty neat, huh?

Problem 4. Prove that

$$f_{n+m} = f_n f_m + f_{n-1} f_{m-1}.$$

(Hint: consider which tilings of the $(n+m)$ -board "break" at the segment separating the n and $(n+1)$ st tiles.) This is especially nice if we take $m = n$,

$$f_{2n} = f_n^2 + f_{n+1} f_{n-1}.$$

Problem 5. Prove

$$f_n = 1 + f_0 + f_1 + \cdots + f_{n-2}$$

by considering the position of the last 2-tile in a tiling of the n -board.

Problem 6. By considering the location of the last 1-tile, prove

$$f_{2n+1} = f_0 + f_2 + \cdots + f_{2n}.$$

and

$$f_{2n} = 1 + f_1 + f_3 + f_5 + \cdots + f_{2n-1}.$$

Problem 7. How does the theory change if, instead of considering tilings by 1- and 2-tiles, we also allow 3-tiles? 4-tiles?

Problem 8. Consider the number of tilings, say $t(k, l)$, of a rectangle of size k -by- l by 1- and 2-tiles. So, for instance, we have seen $t(n, 1) = t(1, n) = f_n$. Find a recursion formula for $t(k, l)$. Can you find any hidden identities here?