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Pell Equations.

Problem 1. Recall the *root-mean-square* of the positive integers $a_1, \dots a_n$ is defined by

$$rms(a_1, \dots, a_n) = \sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}}. \tag{1}$$

What is the smallest positive integer $n \geq 2$ such that $rms(1, 2, \dots, n)$ is an integer?

Solution. Recall the formula for the sum of squares of the first n numbers:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2}$$

Then the problem asks when is

$$rms(1, 2, \dots, n) = \sqrt{\frac{1^2 + 2^2 + \cdots + n^2}{n}} = \sqrt{\frac{n(n+1)(2n+1)}{6n}} = \sqrt{\frac{(n+1)(2n+1)}{6}} \tag{3}$$

a positive integer, say k. In other words, we need to solve the equation

$$\frac{(n+1)(2n+1)}{6} = k^2, \text{ or, equivalently } 2n^2 + 3n + 1 - 6k^2 = 0, \tag{4}$$

for n and k positive integers. Multiply the equation by 8 and complete the square to obtain

$$(4n+3)^2 - 48k^2 = 1. \tag{5}$$

Let us denote $x = 4n+3$ and $y = 4k$. Then we are looking for integer solutions for the equation

$$x^2 - 3y^2 = 1. \tag{6}$$

(in fact x and y have to satisfy the additional requirements that x have residue 3 when divided by 4, and y be divisible by 4.

The equation (6) is a particular case of the so-called (by Euler) *Pell equations*. The more general form is

$$x^2 - Dy^2 = 1, \tag{7}$$

where D is a positive integer which is not a square. The positive integer solutions of these equations are obtained as follows:

1 Find a minimal solution. There is a general algorithm for this involving continued fractions, but our equation is simple enough so we can just guess it. Call this solution (x_0, y_0). In the case of (6) $(x_0, y_0) = (2, 1)$.

2 All the other solutions (x_m, y_m) are obtained from (x_0, y_0) by the formula

$$x_m + y_m\sqrt{D} = (x_0 + y_0\sqrt{D})^{m+1}, n \geq 1. \tag{8}$$

Equation (8) gives also a recursive way to compute (x_m, y_m). Assume (x_{m-1}, y_{m-1}) is known. Then

$$x_m = x_0 x_{m-1} + D y_0 y_{m-1}, \tag{9}$$

$$y_m = y_0 x_{m-1} + x_0 y_{m-1}. \tag{10}$$

In the case of (6), this becomes

$$x_m = 2x_{m-1} + 3y_{m-1},$$
$$y_m = x_{m-1} + 2y_{m-1}.$$

We apply these formulas recusively starting with $(x_0, y_0) = (2, 1)$, and look for the first $m \geq 1$, for which x_m and y_m satisfy the desired conditions for divisibility by 4. We get

m	x_m	y_m
0	2	1
1	7	4
2	26	15
3	97	56
4	362	209
5	1351	780

The first good $m \geq 1$ is $m = 5$. We get $x_m = 1351 = 4n + 3$, so $n = 337$, and $y_m = 780 = 4k$, so $k = 195$.

The answer is $\mathbf{n = 337}$.

Here are some other problems using Pell equations. They are taken from Andrica and Gelca's "Mathematical Olympiad Challenges".

Problem 2. Assume you have ℓ pennies. What are the first three smallest ℓ's for which you can arrange the pennies both in an equilateral triangle, and in a square.

In other words, what are the three smallest ℓ's such that ℓ is both a triangular number $\frac{n(n+1)}{2}$, and a perfect square m^2.

Answer. $\ell = 1,\ 36,\ 1225$. The Pell equation that one obtains is $x^2 - 2y^2 = 1$, with $x = 2n + 1$ and $y = 2m$.

Problem 3. Solve the equation $(x + 1)^3 - x^3 = y^2$ in positive integers.

Problem 4. The triangle with sides $3, 4, 5$ has integer area. Find all triangles with consecutive sides $n - 1, n, n + 1$ and integer area.

(*Hint*: Use Hero's formula of area A in terms of sides a, b, c:

$$A = \frac{1}{4}\sqrt{(a + b + c)(a + b - c)(a + c - b)(b + c - a)}.$$

Problem 5. Find all positive integers m, such that $\frac{m(m+1)}{3}$ is a perfect square.