MATH CIRCLE CONTEST II

December 10, 2003

## STOP!

The city sign maker is trying to figure out how much red paint to buy in order to make some new stop signs. He needs to figure out the area of a regular octagon with edge length 1 foot. What is it? (You may leave factors of $\sqrt{2}$ is your answer.)

Here are some number that you might find useful

$$\tan\left(\frac{2\pi}{8}\right) = \tan(45) = 1$$

$$\tan\left(\frac{2\pi}{16}\right) = \tan(22.5) = \sqrt{2} - 1$$

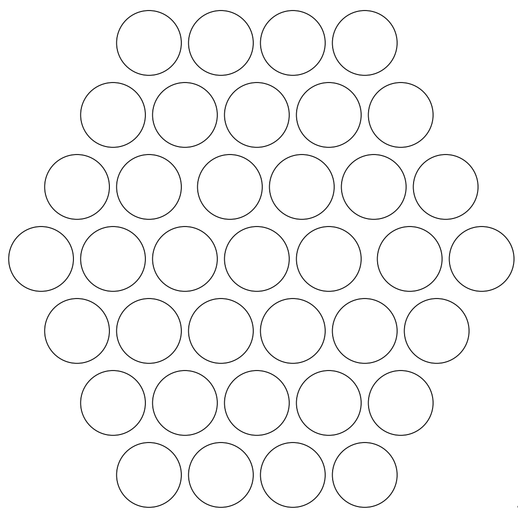
$$\sin\left(\frac{2\pi}{8}\right) = \sin(45) = \cos\left(\frac{2\pi}{8}\right) = \cos(45) = \sqrt{2}/2$$

$$\cos\left(\frac{2\pi}{16}\right) = \cos(22.5) = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\sin\left(\frac{2\pi}{16}\right) = \sin(22.5) = \frac{\sqrt{2-\sqrt{2}}}{2}$$

PRODUCE PYRAMIDS PART TWO

The head grocer are Smiths4D is creating a display of fouranges, a traditional holiday fruit in his four-dimensional homeland. The shadow of a fourange on a three-dimensional wall resembles an earth orange (but the actual taste of a fourange is infinitely more subtle and complex). His display consists of 100 levels which are stacked on top of each other. The three-dimensional shadow of the $k$th level, call if $C_k$, is a kind of three-dimensional pyramid. The three dimensional shadow $C_k$ itself consists of $k$ levels. The two dimensional shadow of the $i$th level in $C_k$, say $D_i$, looks like a regular hexagon with edge length $i$; for instance when $i = 4$, $D_4$ looks like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How many fouranges are in his display?

You may find the following formulae useful:

$$1^2 + 2^2 + \cdots + k^2 = \frac{(k)(k+1)(2k+1)}{6}$$

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$$

$$1^4 + 2^4 + \cdots + k^4 = \frac{6k^5 + 15k^4 + 10k^3 - 1}{30}$$

Fix a vertex $v$ of a four dimensional cube $C$.

(a) How many symmetries of $C$ are there? Justify your answer.

(b) How many symmetries of $C$ fix $v$? Justify your answer.

(Recall that a symmetry of $C$ is a composition of rotations and reflections that leaves $C$ fixed.)